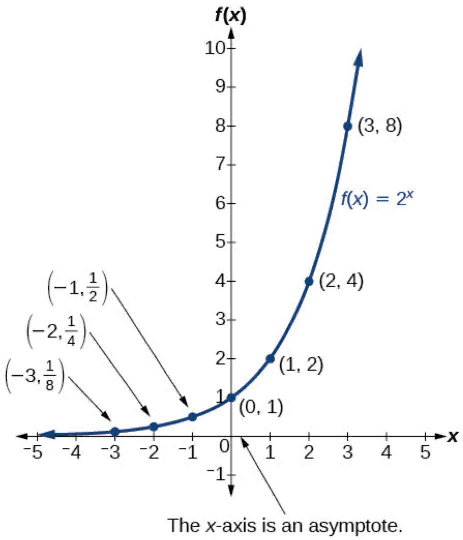
6.2 – Graphs of Exponential Functions

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



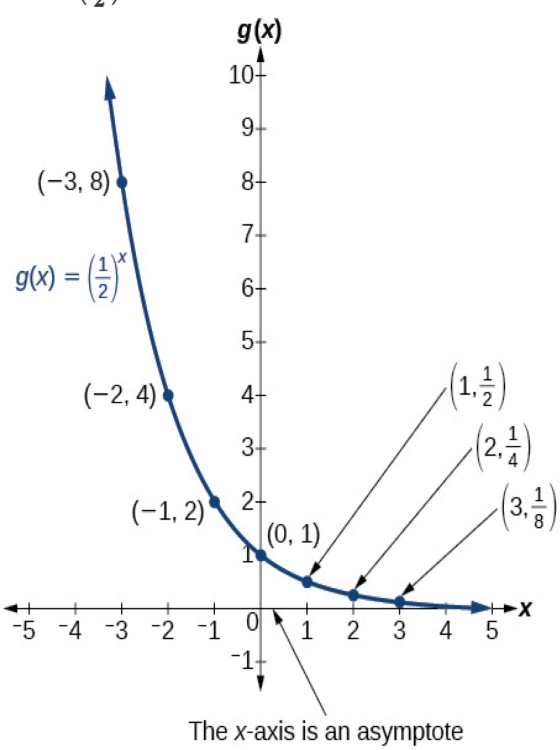
Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of **exponential decay**, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

A GENERAL NOTE: CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $F(X) = B^X$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has these characteristics:

- **one-to-one** function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure compares the graphs of **exponential growth** and decay functions.

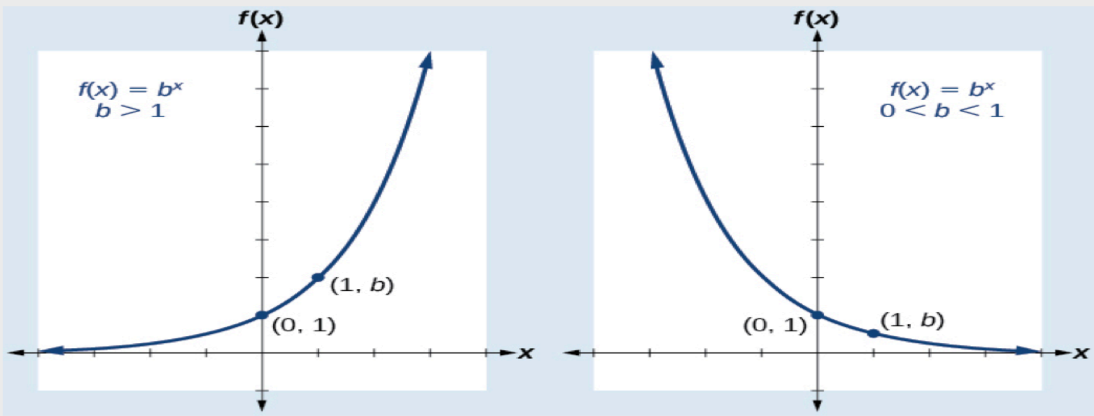


Figure 3.

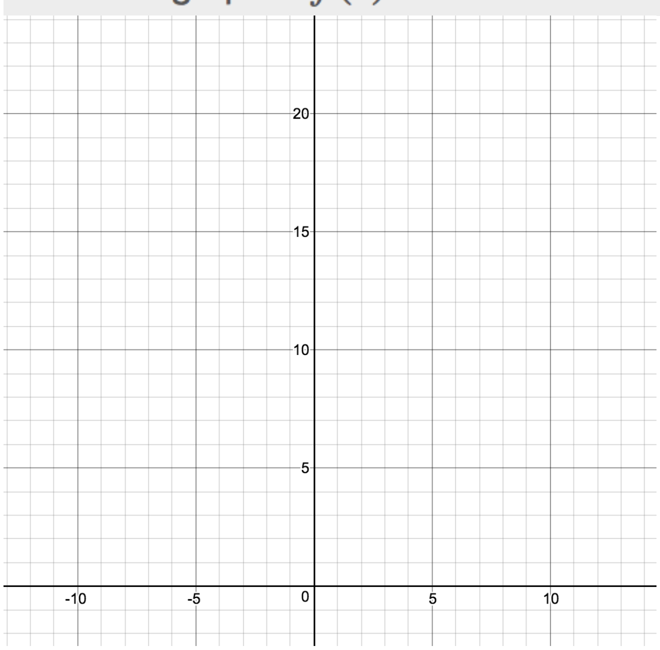
HOW TO

Given an exponential function of the form $f(x) = b^x$, graph the function.

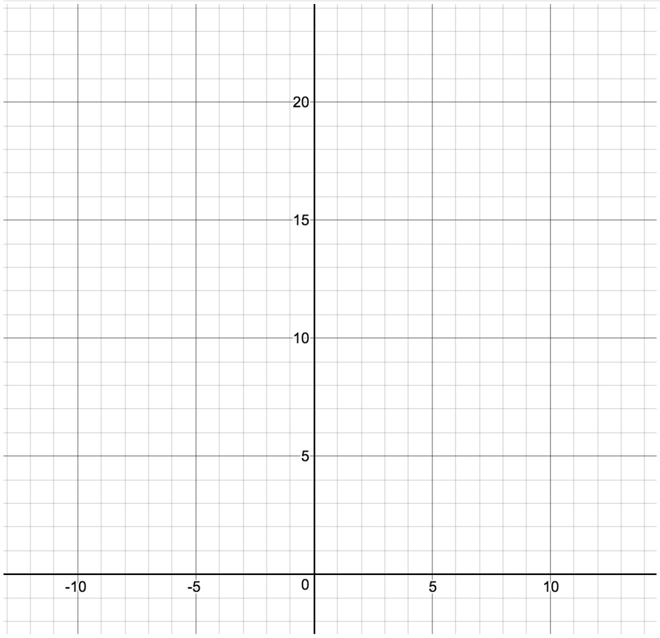
11. Create a table of points.
22. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
33. Draw a smooth curve through the points.
44. State the domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Example

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.



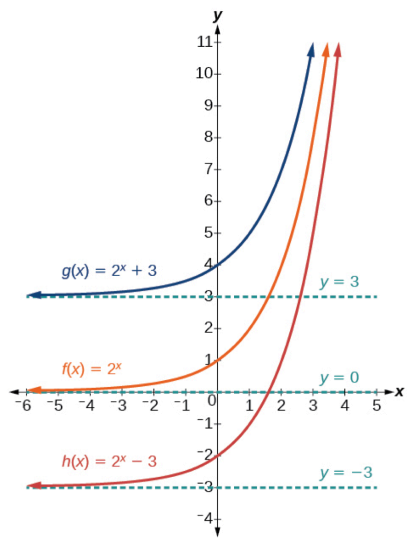
Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.



Graphing Transformations of Exponential Functions

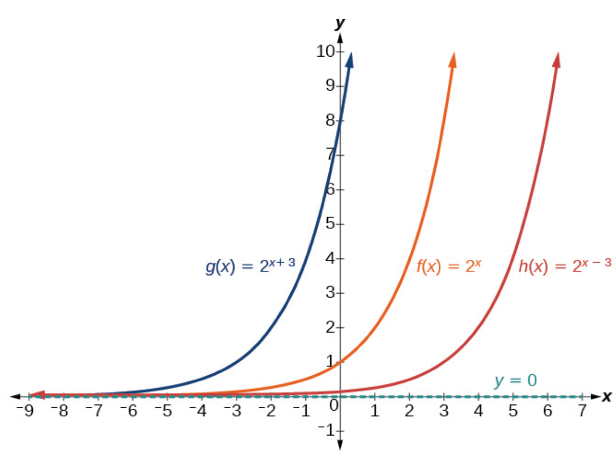
Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.



Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$,the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)\,2^x$,so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$,the y-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right) 2^x$,so the initial value of the function is $\frac{1}{8}$.



A GENERAL NOTE: SHIFTS OF THE PARENT FUNCTION $F(X) = B^X$

For any constants c and d,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$,remains unchanged.

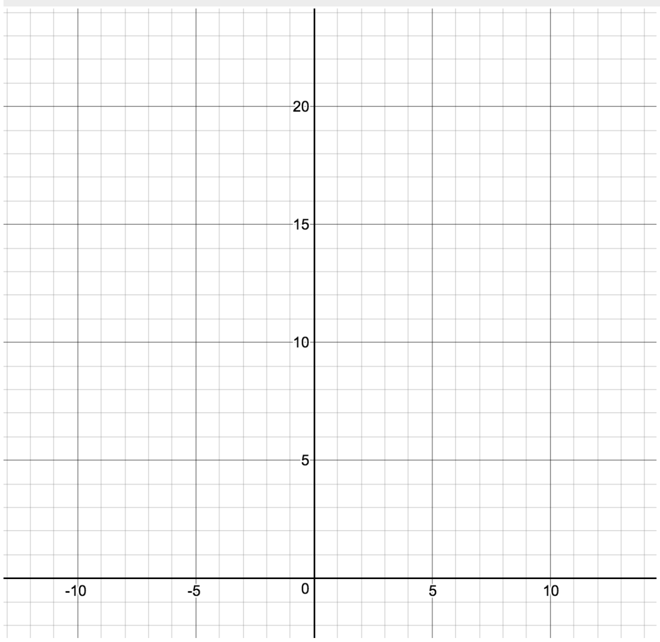
HOW TO

Given an exponential function with the form $f(x) = b^{x+c} + d$,graph the translation.

1.1. Draw the horizontal asymptote $y = d$.
22. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
33. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
44. State the domain, $(-\infty, \infty)$,the range, (d, ∞),and the horizontal asymptote $y = d$.

Example

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.



HOW TO

Given an equation of the form $f(x) = b^{x+c} + d$ for x, use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y_1=**".
- Enter the given value for $f(x)$ in the line headed "**Y_2=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**Y_2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.
- To find the value of x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Examples

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Graphing a Stretch or Compression

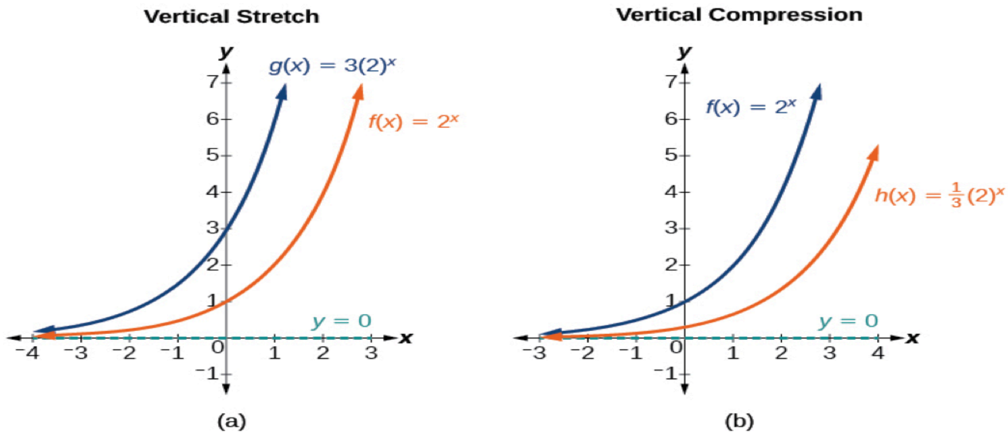


Figure 6. (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

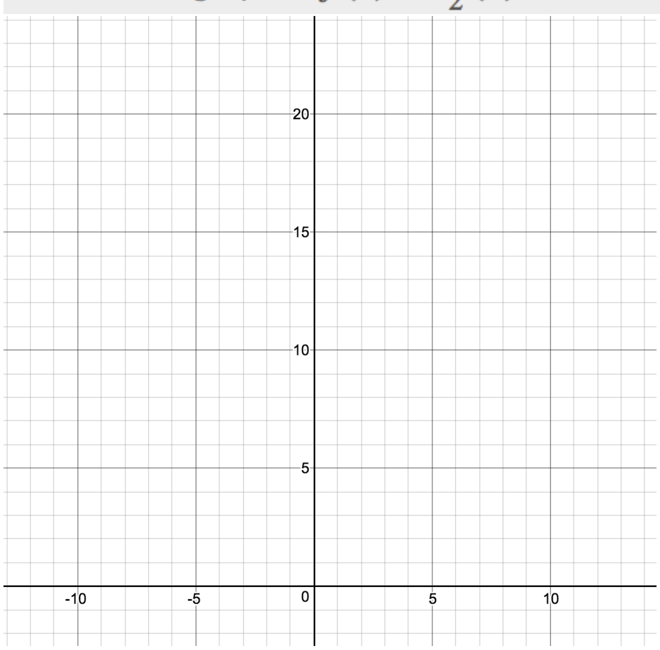
A GENERAL NOTE: STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $F(X) = B^X$

For any factor $a > 0$, the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Example

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.



Graphing Reflections

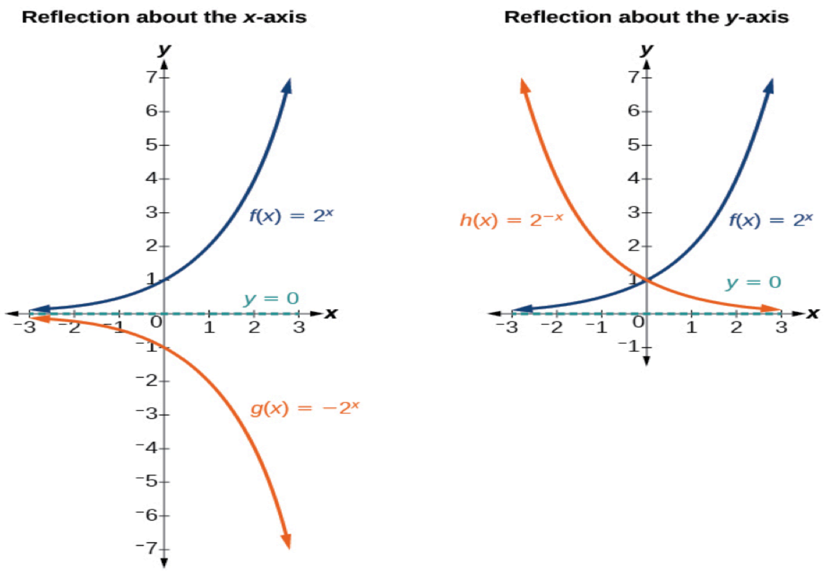


Figure 8. (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

A GENERAL NOTE: REFLECTIONS OF THE PARENT FUNCTION $F(X) = B^X$

The function $f(x) = -b^x$

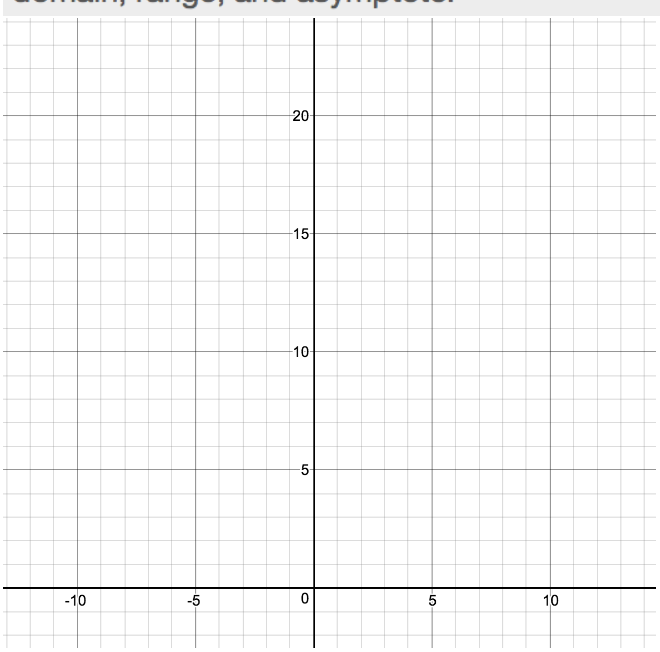
- reflects the parent function $f(x) = b^x$ about the x-axis.
- has a y-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

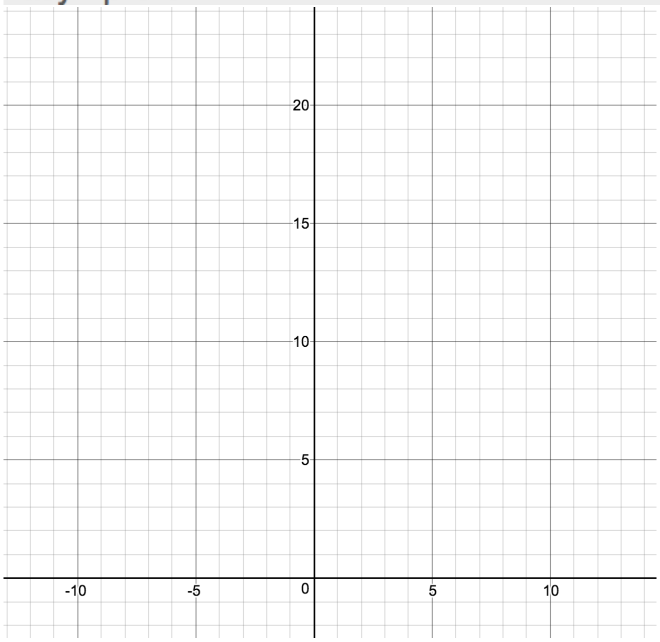
- reflects the parent function $f(x) = b^x$ about the y-axis.
- has a y-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Examples

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.



Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.



Summarizing Transformations of Exponential Functions

Translations of the Parent Function $f(x) = b^x$

Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $0 < \lvert a \rvert < 1$	$f(x) = ab^x$
Reflect about the *x*-axis	$f(x) = -b^x$
Reflect about the *y*-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

A GENERAL NOTE: TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x, b > 1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the x-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Examples

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.