

5.6 – Rational Functions

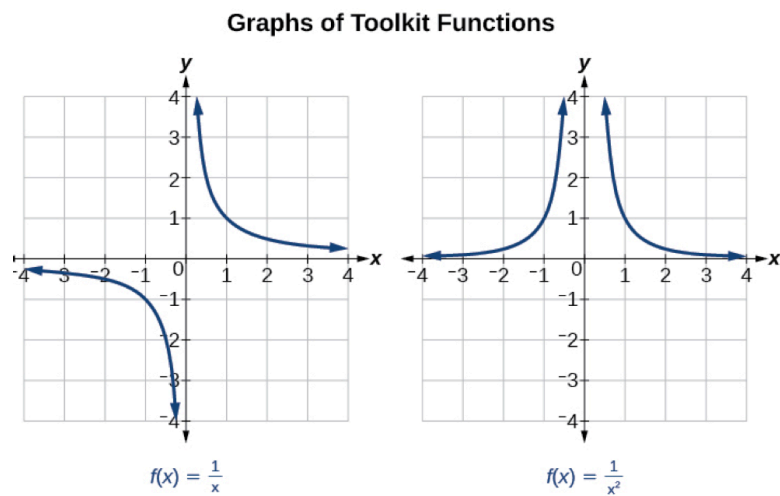
Suppose we know that the cost of making a product is dependent on the number of items, x , produced. This is given by the equation $C(x)=15,000x-0.1x^2+1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x .

The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

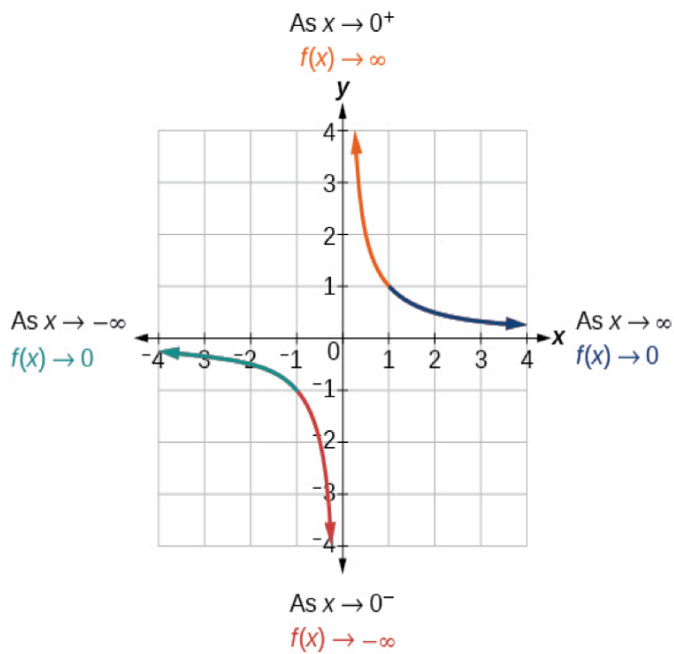
Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power. In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation



Symbol	Meaning
$x \rightarrow a^-$	x approaches a from the left ($x < a$ but close to a)
$x \rightarrow a^+$	x approaches a from the right ($x > a$ but close to a)
$x \rightarrow \infty$	x approaches infinity (x increases without bound)
$x \rightarrow -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \rightarrow \infty$	the output approaches infinity (the output increases without bound)
$f(x) \rightarrow -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \rightarrow a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$ & End Behavior of $f(x) = \frac{1}{x}$



This behavior creates a **vertical**

_____, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x=0$ as the input becomes close to zero.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a _____

asymptote, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y=0$.

A GENERAL NOTE: VERTICAL ASYMPTOTE

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a . We write

$$\text{As } x \rightarrow a, f(x) \rightarrow \infty, \text{ or as } x \rightarrow a, f(x) \rightarrow -\infty.$$

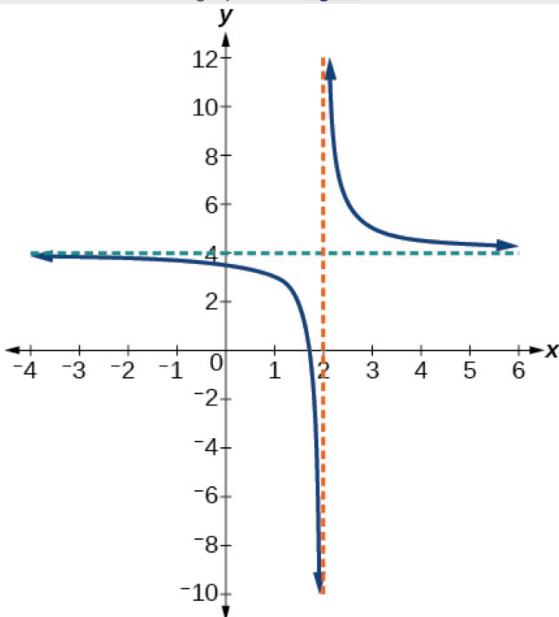
A GENERAL NOTE: HORIZONTAL ASYMPTOTE

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

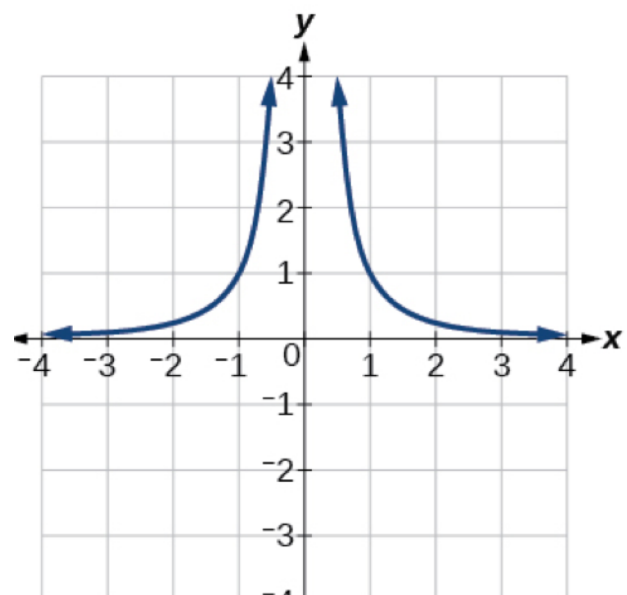
$$\text{As } x \rightarrow \infty \text{ or } x \rightarrow -\infty, f(x) \rightarrow b.$$

Examples

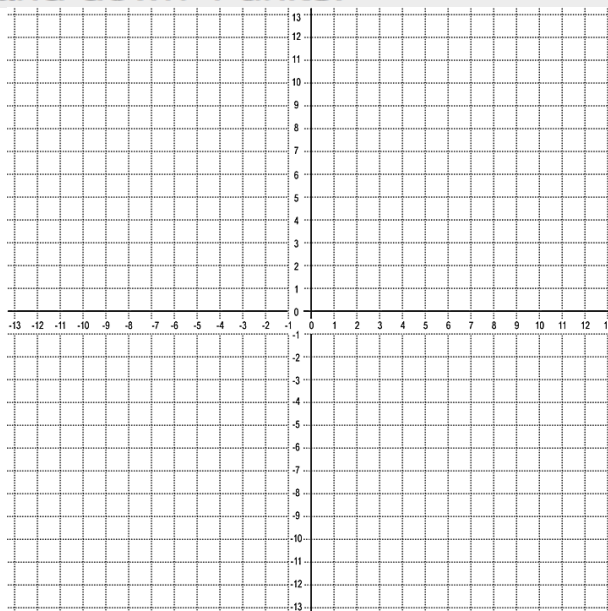
Use arrow notation to describe the end behavior and local behavior of the function graphed in [Figure](#).



Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.



Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.



Solving Applied Problems Involving Rational Functions

A GENERAL NOTE: RATIONAL FUNCTION

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_px^p + a_{p-1}x^{p-1} + \dots + a_1x + a_0}{b_qx^q + b_{q-1}x^{q-1} + \dots + b_1x + b_0}, Q(x) \neq 0$$

Examples

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Finding the Domains of Rational Functions

A GENERAL NOTE: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

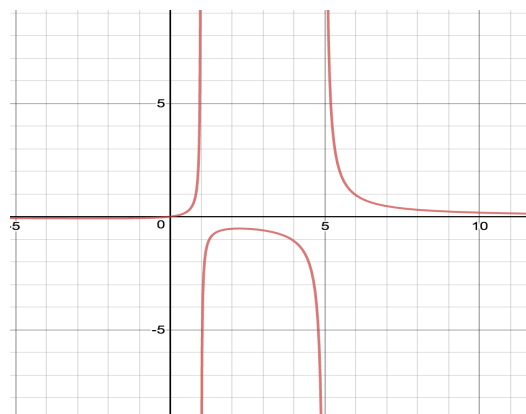
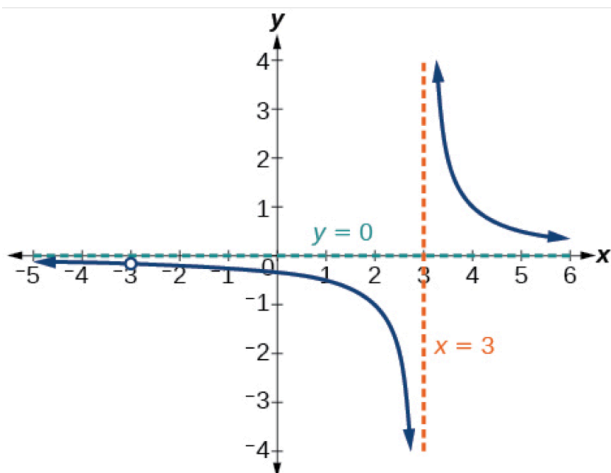
Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x -values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Examples

Find the domain of $f(x) = \frac{x+3}{x^2-9}$

Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$



Identifying Vertical Asymptotes of Rational Functions

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

HOW TO

Given a rational function, identify any vertical asymptotes of its graph.

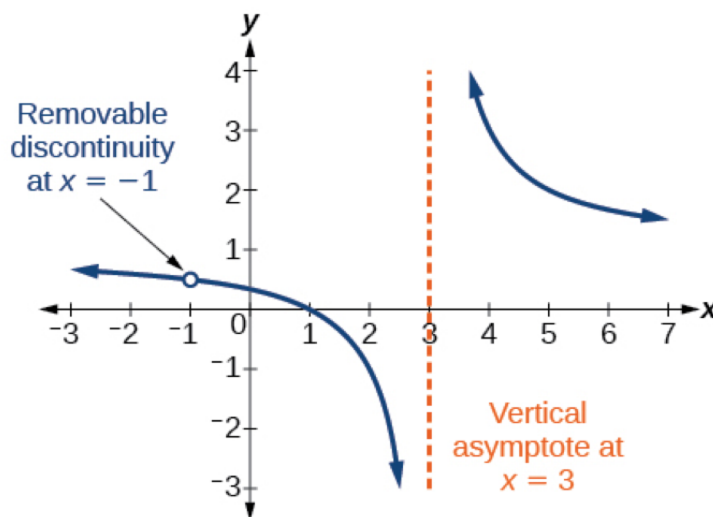
1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

Example

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Removable Discontinuities

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$



A GENERAL NOTE: REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Examples

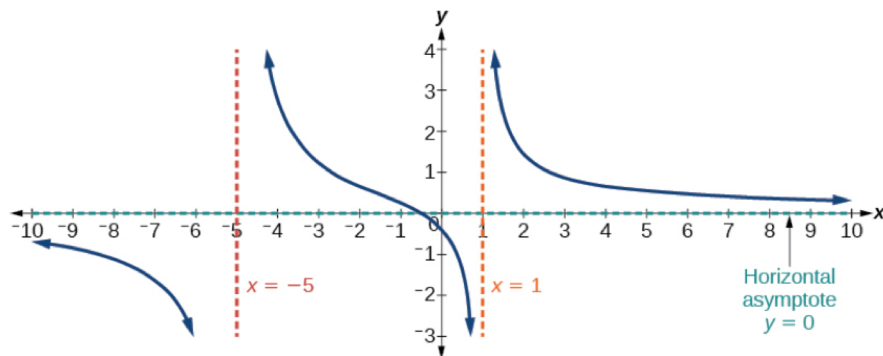
Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying Horizontal Asymptotes of Rational Functions

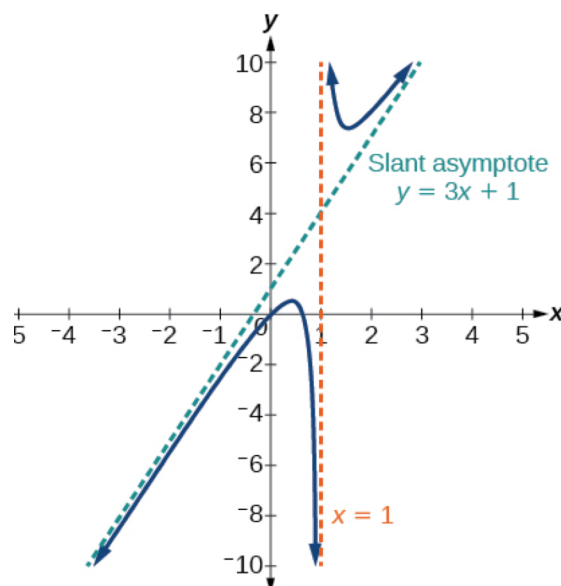
Case 1: If the degree of the denominator $>$ degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

Example: $f(x) = \frac{4x + 2}{x^2 + 4x - 5}$



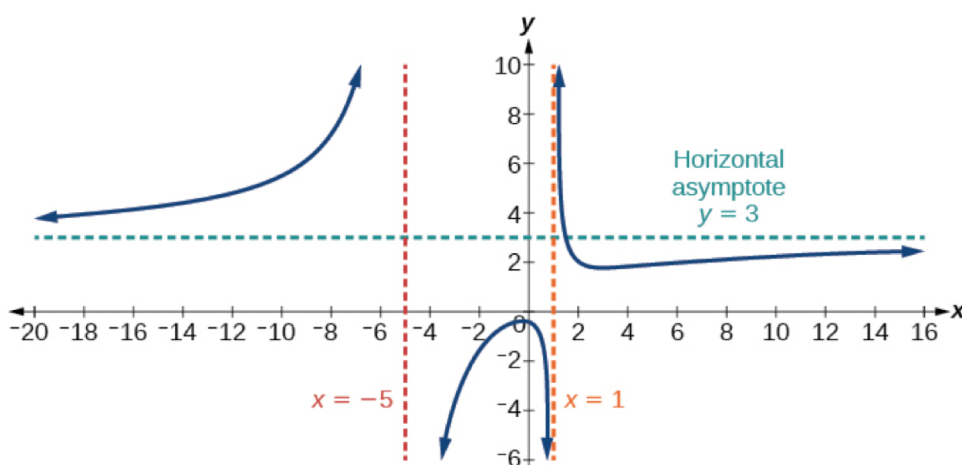
Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x) = \frac{3x^2 - 2x + 1}{x - 1}$



Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

Example: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$



Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote. It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction.

A GENERAL NOTE: HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

Examples

For the functions listed, identify the horizontal or slant asymptote.

a. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$

b. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$

c. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Intercepts of Rational Functions

A GENERAL NOTE: INTERCEPTS OF RATIONAL FUNCTIONS

A **rational function** will have a y -intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y -intercept if the function is not defined at zero.

Likewise, a rational function will have x -intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x -intercepts can only occur when the numerator of the rational function is equal to zero.

Examples

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x - and y -intercepts and the horizontal and vertical asymptotes.

Graphing Rational Functions

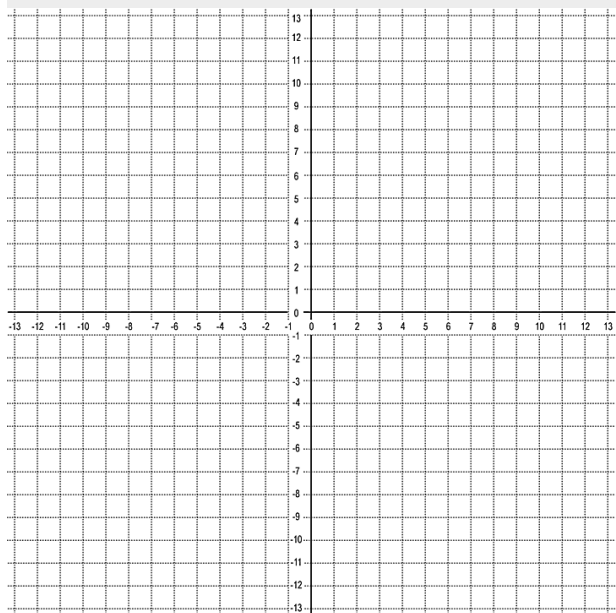
HOW TO

Given a rational function, sketch a graph.

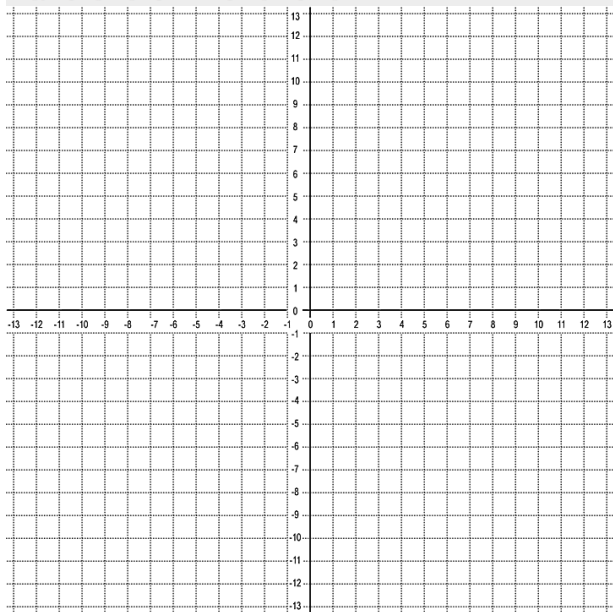
1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Examples

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.



Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.



Writing Rational Functions

A GENERAL NOTE: WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x -intercepts at $x = x_1, x_2, \dots, x_n$, vertical asymptotes at $x = v_1, v_2, \dots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form:

$$f(x) = a \frac{(x - x_1)^{p_1} (x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1} (x - v_2)^{q_2} \cdots (x - v_m)^{q_m}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x -intercept or by the horizontal asymptote if it is nonzero.

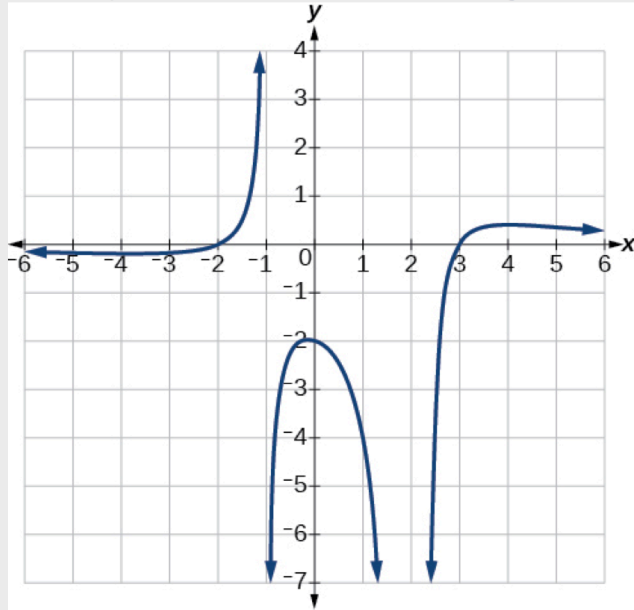
HOW TO

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the x -intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the “simplest” function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Examples

Write an equation for the rational function shown in [Figure](#).



Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$