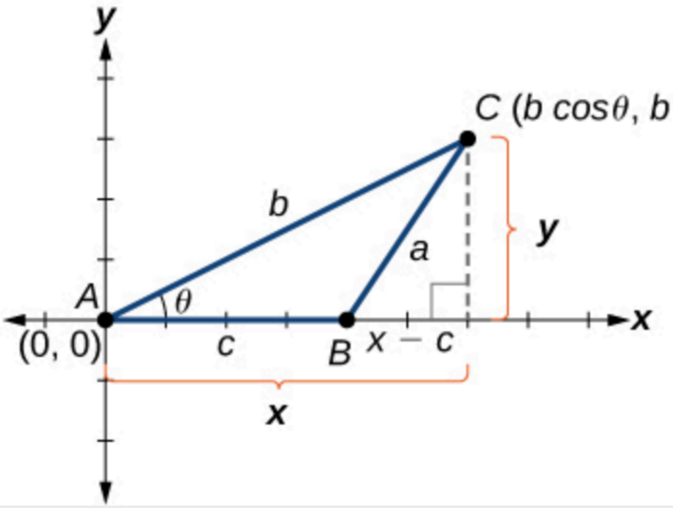
Using the Law of Cosines to Solve Oblique Triangles

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle *ABC* is placed in the coordinate plane with vertex *A* at the origin, side *c* drawn along the *x*-axis, and vertex *C* located at some point(*x*,*y*)in the plane, as illustrated in Figure. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.



A GENERAL NOTE: LAW OF COSINES

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in Figure, with angles $\alpha, \beta,$ and γ, and opposite corresponding sides $a, b,$ and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta$$
$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

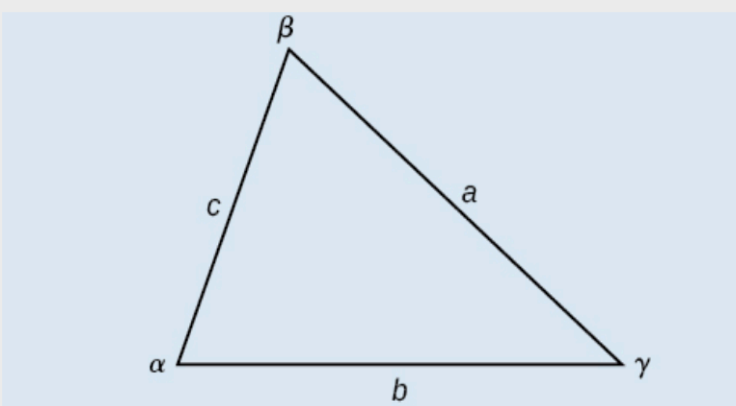


Figure 3.

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

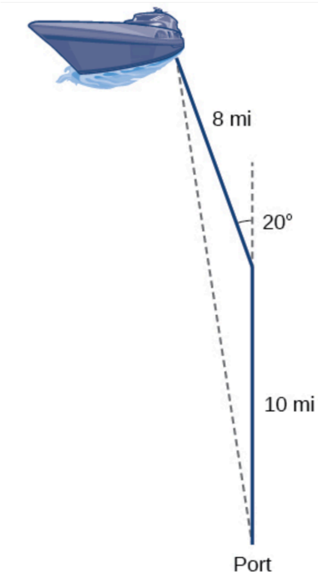
$$\cos \alpha = \frac{b^2+c^2-a^2}{2bc}$$
$$\cos \beta = \frac{a^2+c^2-b^2}{2ac}$$
$$\cos \gamma = \frac{a^2+b^2-c^2}{2ab}$$

Examples

Find the missing side and angles of the given triangle: $\alpha = 30°,\ b = 12,\ c = 24.$

Given $a = 5, b = 7,$ and $c = 10,$ find the missing angles.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure. How far from port is the boat?



Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron's formula instead of finding the height.

A GENERAL NOTE: HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b, and c are known.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

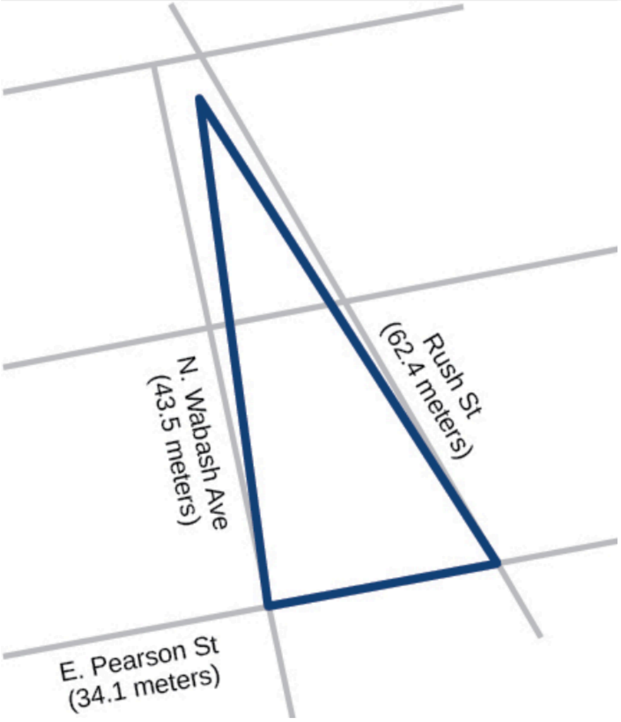
where $s = \frac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure for a view of the city property.



Find the area of a triangle given $a = 4.38$ ft, $b = 3.79$ ft, and $c = 5.22$ ft.