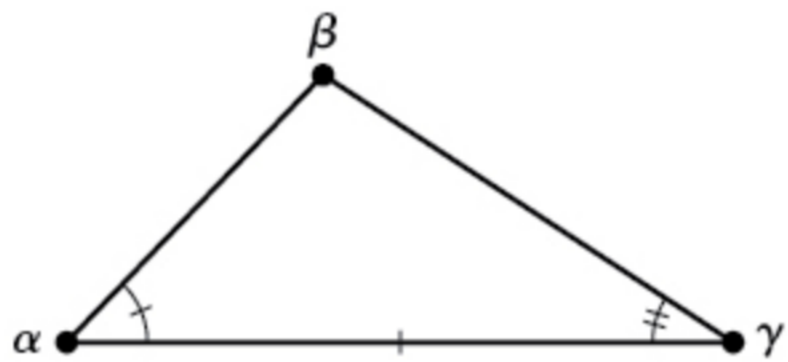
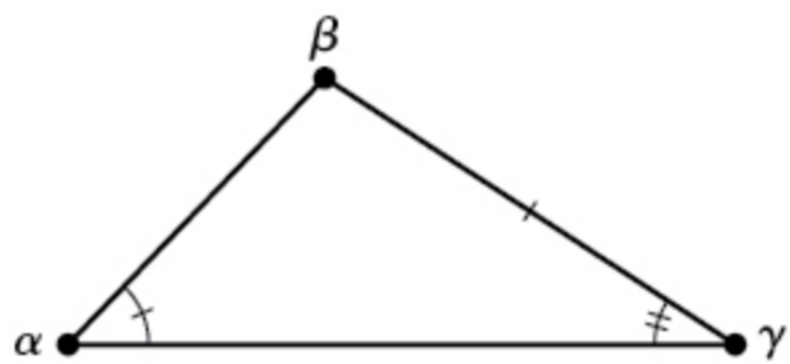
10.1 – Non-Right Triangles: The Law of Sines

Using Law of Sines to Solve Oblique Triangles
Any triangle that is not a right triangle is an ________________________ triangle. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

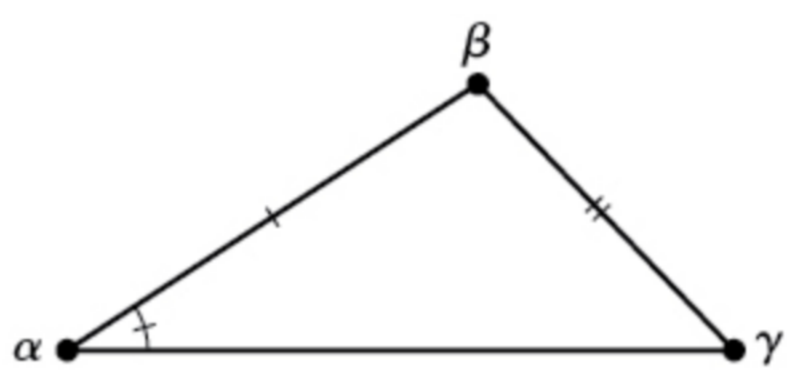
1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side.



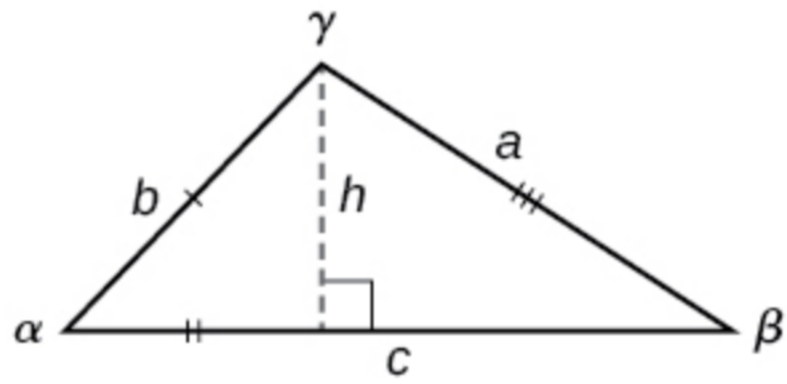
2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles.



3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides.



How do right triangle relationships set up the Law of Sines?



A GENERAL NOTE: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

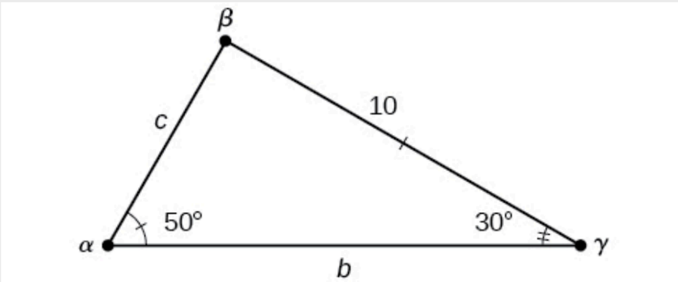
$$\frac{\sin\ \alpha}{a} = \frac{\sin\ \beta}{b} = \frac{\sin\ \gamma}{c}$$

$$\frac{a}{\sin\ \alpha} = \frac{b}{\sin\ \beta} = \frac{c}{\sin\ \gamma}$$

To solve an oblique triangle, use any pair of applicable ratios.

Examples

Solve the triangle shown in Figure to the nearest tenth.



Solve the triangle shown in Figure to the nearest tenth.

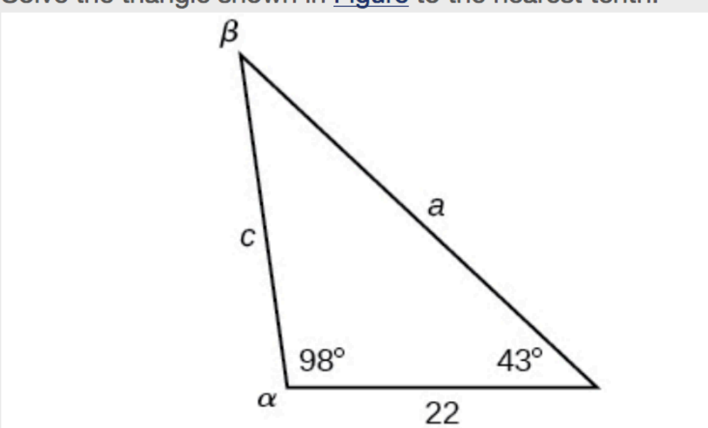


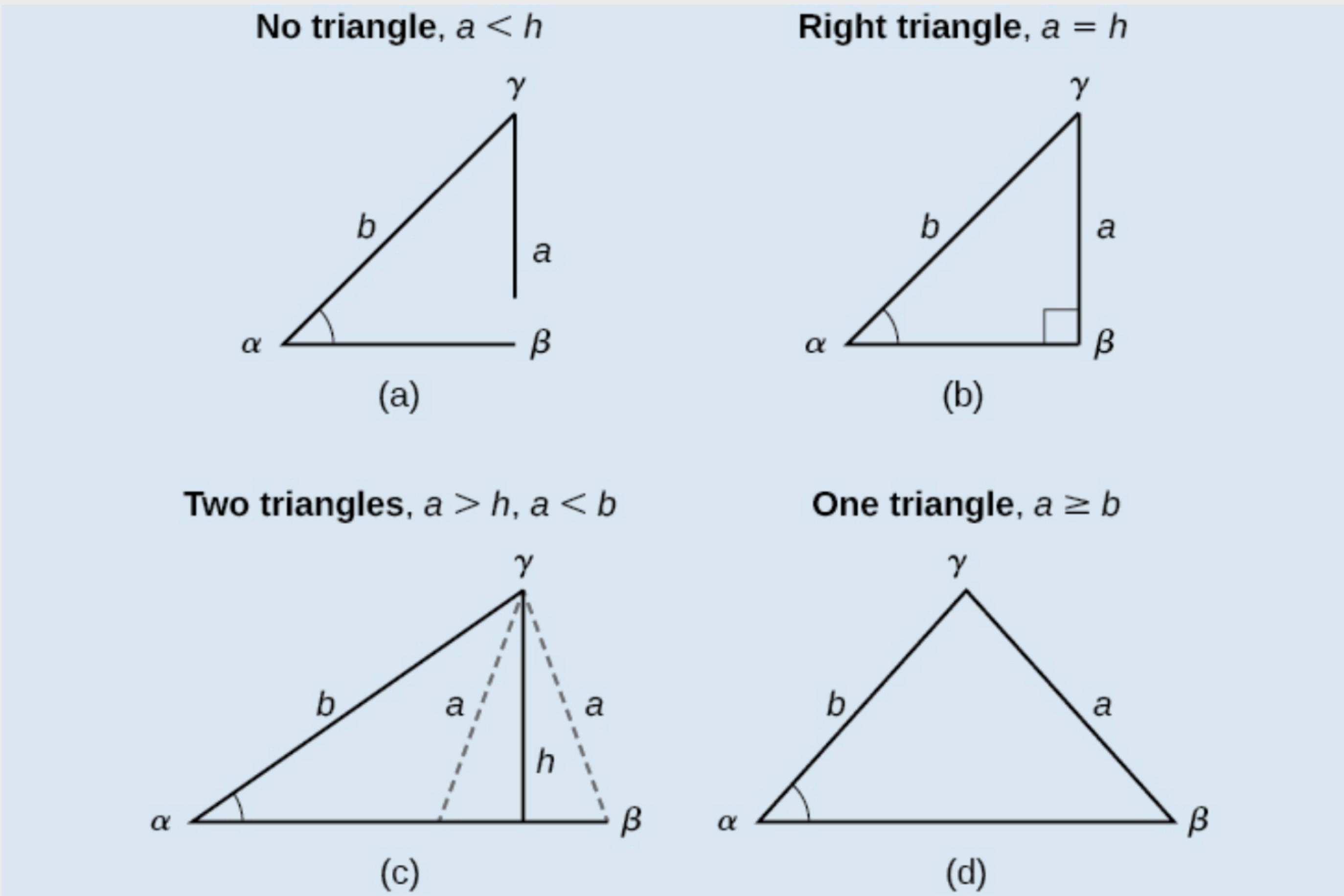
Figure 8.

Using the Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ______________________________________ case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

A GENERAL NOTE: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure illustrates the solutions with the known sides a and b and known angle α.



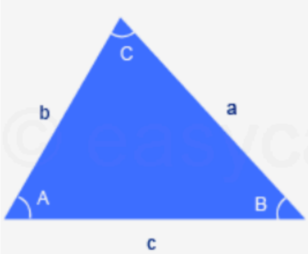
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Equations from Law of Sines solving for angles A,B and C

$$A = \sin^{-1}\left[\frac{a \sin B}{b}\right] \qquad A = \sin^{-1}\left[\frac{a \sin C}{c}\right]$$

$$B = \sin^{-1}\left[\frac{b \sin A}{a}\right] \qquad B = \sin^{-1}\left[\frac{b \sin C}{c}\right]$$

$$C = \sin^{-1}\left[\frac{c \sin A}{a}\right] \qquad C = \sin^{-1}\left[\frac{c \sin B}{b}\right]$$



Equations from Law of Sines solving for sides a,b and c

Equations from Law of Sines solving for sides a,b and c

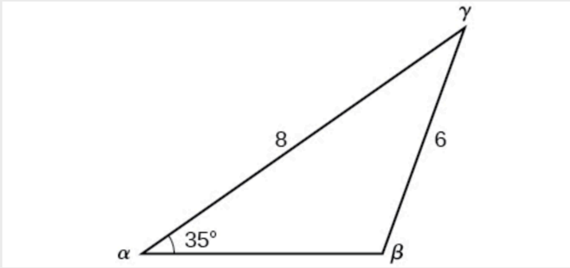
$$a = \frac{b \sin A}{\sin B} \qquad a = \frac{c \sin A}{\sin C}$$

$$b = \frac{a \sin B}{\sin A} \qquad b = \frac{c \sin B}{\sin C}$$

$$c = \frac{a \sin C}{\sin A} \qquad c = \frac{b \sin C}{\sin B}$$

Examples

Solve the triangle in Figure for the missing side and find the missing angle measures to the nearest tenth.



In the triangle shown in Figure, solve for the unknown side and angles. Round your answers to the nearest tenth.

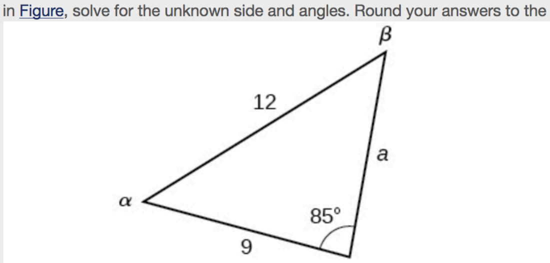
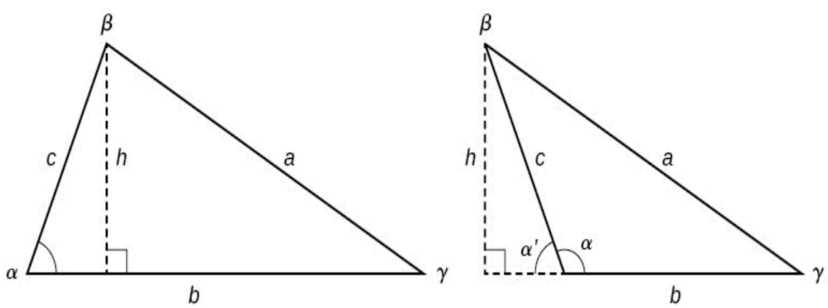


Figure 13.

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Finding the Area of an Oblique Triangle Using the Sine Function

Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $sin\alpha = \frac{h}{c}$ or $c \sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.



$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}b(c\sin\alpha)$$

$$\text{Area} = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

A GENERAL NOTE: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}bc\sin\alpha \\ &= \tfrac{1}{2}ac\sin\beta \\ &= \tfrac{1}{2}ab\sin\gamma\end{aligned}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Examples

Find the area of a triangle with sides $a = 90, b = 52,$ and angle $\gamma = 102°$. Round the area to the nearest integer.

Find the area of the triangle given $\beta = 42°,\ a = 7.2\,\text{ft},\ c = 3.4\,\text{ft}.$ Round the area to the nearest tenth.

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure. Round the altitude to the nearest tenth of a mile.