9.4 – Sum-to-Product and Product-to-Sum

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for cosine. If we add the two equations, we get:

$$\begin{array}{rrcl} & \cos\ \alpha\ \cos\ \beta + \sin\ \alpha\ \sin\ \beta & = & \cos(\alpha - \beta) \\ + & \cos\ \alpha\ \cos\ \beta - \sin\ \alpha\ \sin\ \beta & = & \cos(\alpha + \beta) \\ \hline \end{array}$$

Examples

Write the following product of cosines as a sum: $2\ \cos\left(\frac{7x}{2}\right)\ \cos\ \frac{3x}{2}$.

Use the product-to-sum formula to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for sine. If we add the sum and difference identities, we get:

$$\begin{array}{rrcl} & \sin(\alpha + \beta) & = & \sin\ \alpha\ \cos\ \beta + \cos\ \alpha\ \sin\ \beta \\ + & \sin(\alpha - \beta) & = & \sin\ \alpha\ \cos\ \beta - \cos\ \alpha\ \sin\ \beta \\ \hline \end{array}$$

Examples

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of cosine is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{array}{rl} & \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ - & \cos(\alpha+\beta) = -(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ \hline & \end{array}$$

A GENERAL NOTE: THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum formulas** are as follows:

$$\begin{aligned} \cos\alpha\cos\beta &= \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \\ \sin\alpha\sin\beta &= \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\ \cos\alpha\sin\beta &= \tfrac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \end{aligned}$$

Examples

Write $\cos(3\theta)\ \cos(5\theta)$ as a sum or difference.

Use the product-to-sum formula to evaluate $\cos\ \frac{11\pi}{12}\ \cos\ \frac{\pi}{12}$.

Expressing Sums as Products

A GENERAL NOTE: SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\begin{aligned}
\sin\ \alpha + \sin\ \beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin\ \alpha - \sin\ \beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\\
\cos\ \alpha - \cos\ \beta &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
\cos\ \alpha + \cos\ \beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

Examples

Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta) + \sin(\theta)$.

Evaluate $\cos(15°) - \cos(75°)$. Check the answer with a graphing calculator.

Prove the identity:

$$\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} = -\tan t$$

Verify the identity $\csc^2\theta - 2 = \frac{\cos(2\theta)}{\sin^2\theta}$.

Verify the identity $\tan\theta \cot\theta - \cos^2\theta = \sin^2\theta$.