9.3 – Double-Angle, Half-Angle, and Reduction Formulas

The __________________ _______________formulas are a special case of the sum formulas, where $\alpha=\beta$. Deriving the double-angle formula for sine begins with the sum formula.

$$\sin(\alpha+\beta)=\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\cos(2\theta) \quad = \quad \cos^2\theta-\sin^2\theta$$

$$\cos(2\theta) \quad = \quad \cos^2\theta-\sin^2\theta$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha=\beta=\vartheta$ in the sum formula gives

$$\tan(\alpha+\beta) \quad = \quad \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

A GENERAL NOTE: DOUBLE-ANGLE FORMULAS

The **double-angle formulas** are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

HOW TO

Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value.

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example

Given $\sin\alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Use Reduction Formulas to Simplify an Expression

The ___________________ ________________ formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine.

Solve for $\sin^2\vartheta$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

Solve for $\cos^2\vartheta$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

A GENERAL NOTE: REDUCTION FORMULAS

The **reduction formulas** are summarized as follows:

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Examples

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Use the power-reducing formulas to prove

$$\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right]\left[1 - \cos(4x)\right]$$

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of __________ _______________ formulas, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin(\frac{\alpha}{2})$. Note that the half-angle formulas are preceded by a ± sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\alpha 2$ terminates.

The half-angle formula for sine is derived as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

To derive the half-angle formula for cosine:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

A GENERAL NOTE: HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\begin{aligned}\tan\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\ &= \frac{\sin\alpha}{1+\cos\alpha}\\ &= \frac{1-\cos\alpha}{\sin\alpha}\end{aligned}$$

HOW TO

Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Examples

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

1. $\sin\left(\frac{\alpha}{2}\right)$
2. $\cos\left(\frac{\alpha}{2}\right)$
3. $\tan\left(\frac{\alpha}{2}\right)$

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.