8.3 – Inverse Trigonometric Functions

Understanding and Using Inverse Sine, Cosine, and Tangent Functions

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

For example, if $f(x) = \sin x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

Recall that, for a one-to-one function, if $f(a)=b$, then an inverse function would satisfy $f^{-1}(\quad)= \quad$. Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the ____________________ of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0.

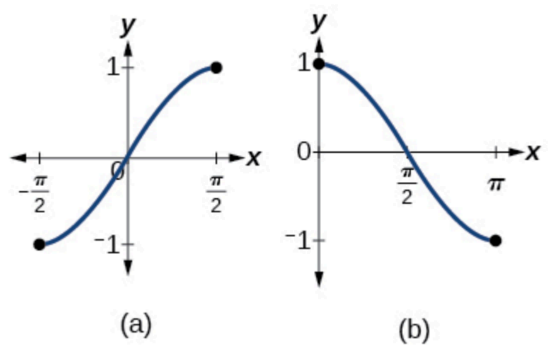


Figure 2. (a) Sine function on a restricted domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

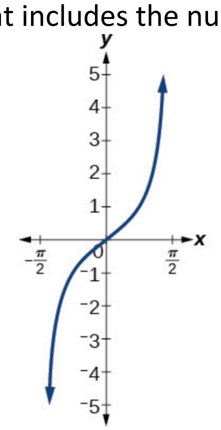


Figure 3. Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

On these restricted domains, we can define the **inverse trigonometric functions**.

- The **inverse sine function** $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

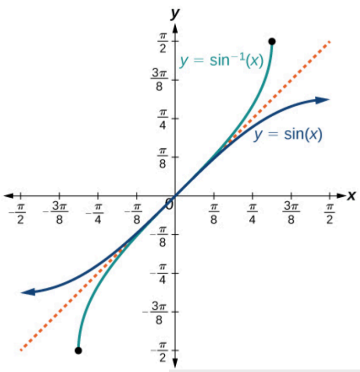
- The **inverse cosine function** $y = \cos^{-1} x$ means $x = \cos y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos x$.

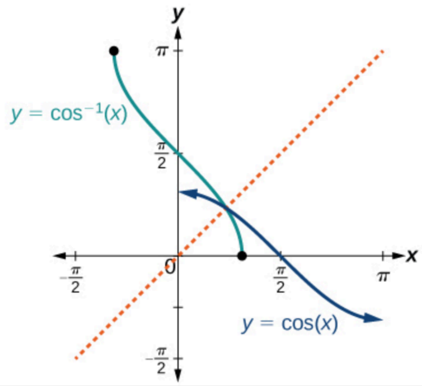
$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

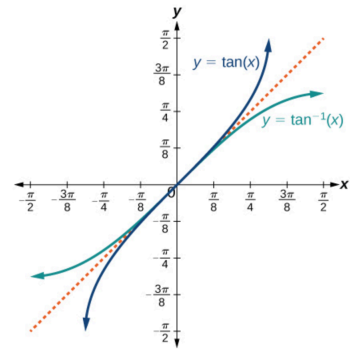
- The **inverse tangent function** $y = \tan^{-1} x$ means $x = \tan y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

To find the domain and range of inverse trigonometric functions, _______________ the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line *y=x*.







RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

Examples

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

HOW TO

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Examples

Evaluate each of the following.

1. $\sin^{-1}\left(\frac{1}{2}\right)$
2. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
3. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
4. $\tan^{-1}(1)$

Evaluate each of the following.

1. $\sin^{-1}(-1)$
2. $\tan^{-1}(-1)$
3. $\cos^{-1}(-1)$
4. $\cos^{-1}\left(\frac{1}{2}\right)$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

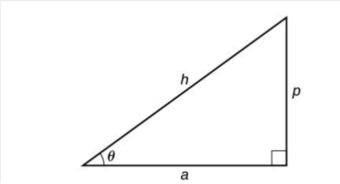
Examples -→ RADIANS!!!!!!

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

HOW TO

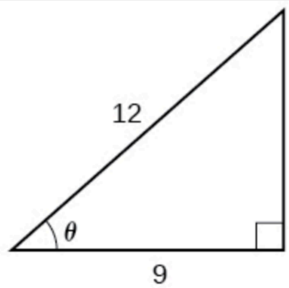
Given two sides of a right triangle like the one shown in Figure, find an angle.



1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Examples

Solve the triangle in Figure for the angle θ.



Solve the triangle in Figure for the angle θ.

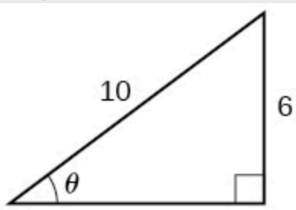


Figure 9.

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

For any trigonometric function, $f(f^{-1}(y))=y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}.However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

A GENERAL NOTE: COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\sin(\sin^{-1}x) = x \text{ for } -1 \le x \le 1$$
$$\cos(\cos^{-1}x) = x \text{ for } -1 \le x \le 1$$
$$\tan(\tan^{-1}x) = x \text{ for } -\infty < x < \infty$$

$$\sin^{-1}(\sin x) = x \text{ only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\cos^{-1}(\cos x) = x \text{ only for } 0 \le x \le \pi$$
$$\tan^{-1}(\tan x) = x \text{ only for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

HOW TO

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta,\ \cos\theta,$ or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Examples

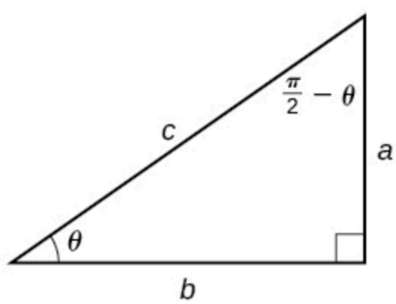
Evaluate the following:

1. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
2. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
3. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
4. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$

Evaluating Compositions of the Form $f^{-1}(g(x))$

For special values of x, we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is ϑ, making the other $\frac{\pi}{2} = \theta$.Consider the sine and cosine of each angle of the right triangle in Figure.



Because $\cos\theta = \frac{b}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

HOW TO

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y$$

Examples

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

1. by direct evaluation.
2. by the method described previously.

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Evaluating Composition of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f\left(g^{-1}(x)\right)$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin\left(\cos^{-1}x\right) = \sqrt{1-x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the **inverse trigonometric functions** to find compositions involving algebraic expressions.

Examples

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \leq x \leq 3$.

Find a simplified expression for $\sin\left(\tan^{-1}(4x)\right)$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$.