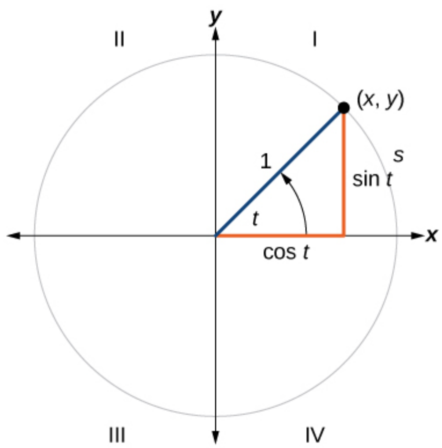
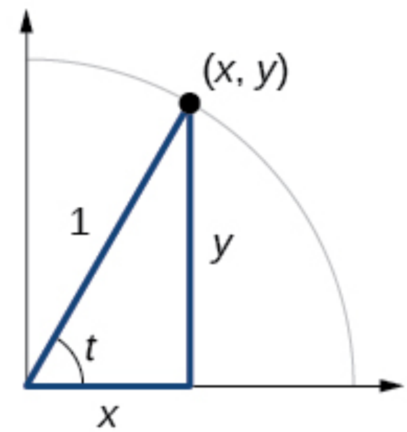
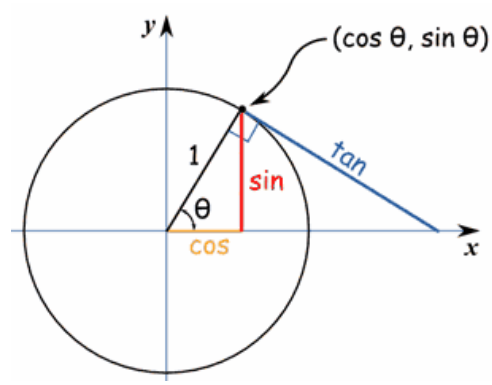
7.3 – The Unit Circle

Finding Trig Functions Using The Unit Circle

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates,(x,y).The coordinates x and y will be the outputs of the trigonometric functions $f(t)=\cos t$ and $f(t)=\sin t$, respectively. This means $x=\cos t$ and $y=\sin t$.







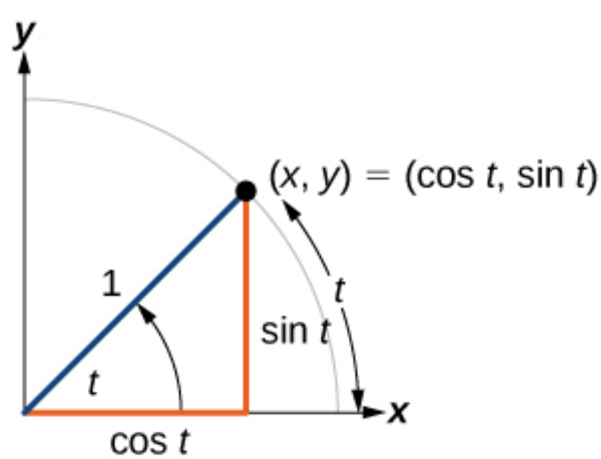
A GENERAL NOTE: UNIT CIRCLE

A **unit circle** has a center at $(0, 0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of **arc length** s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

Like all functions, the sine function has an input and an output. Its input is the measure of the ________________, its output is the ___-coordinate of the corresponding point on the unit circle. The cosine function of an angle t equals the x-value of the endpoint on the unit circle of an arc of _______________ t. In the figure, cosine is equal to _____.



Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise,$\cos^2 t$ is a commonly used shorthand notation for$(\cos(t))^2$.Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

A GENERAL NOTE: SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to a central angle t, then

$$\cos\ t = x$$

$$\sin\ t = y$$

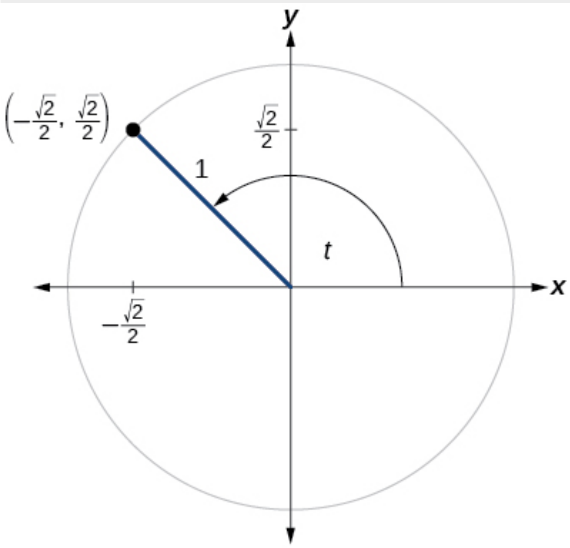
HOW TO

Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine.

1. The sine of t is equal to the y-coordinate of point P : $\sin\ t = y$.
2. The cosine of t is equal to the x-coordinate of point P : $\cos\ t = x$.

Example

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure. Find $\cos\ t$ and $\sin\ t$.

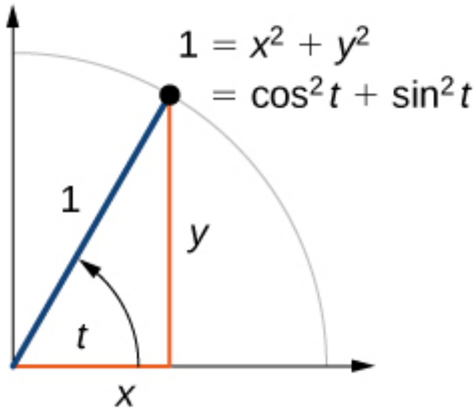


Examples

Find $\cos(90°)$ and $\sin(90°)$.

Find cosine and sine of the angle π.

The Pythagorean Identity



A GENERAL NOTE: PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1$$

HOW TO

Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin\ t$ into the Pythagorean Identity.
2. Solve for $\cos\ t$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

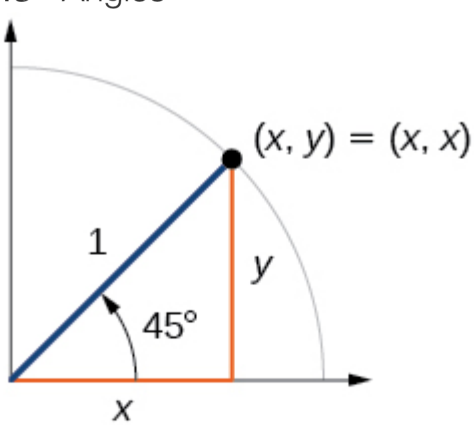
Examples

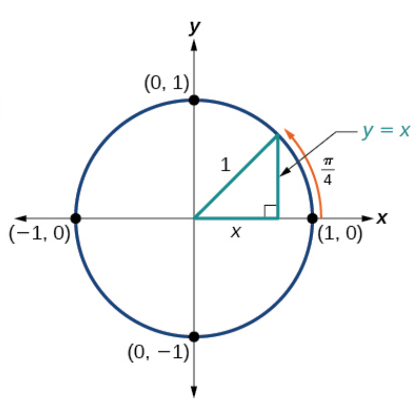
If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

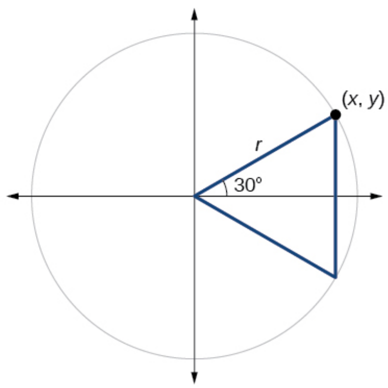
Finding Sine and Cosine of Special Angles

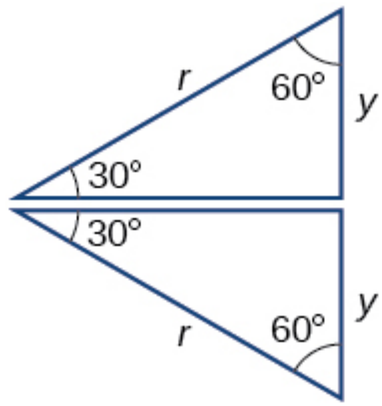
45° Angles

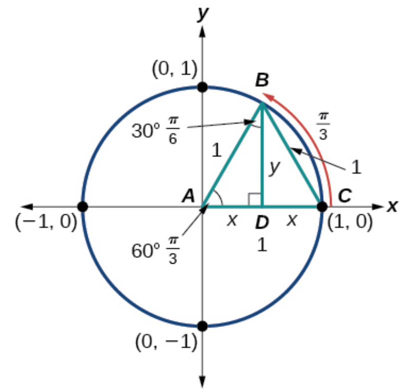




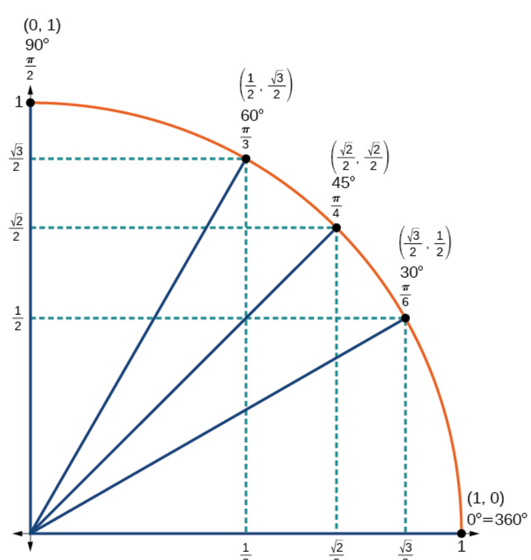
$30°$ and $60°$ Angles

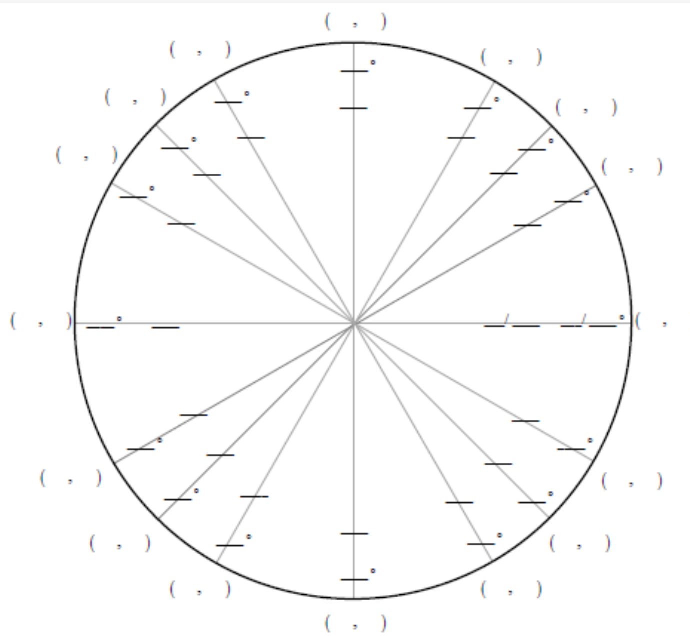




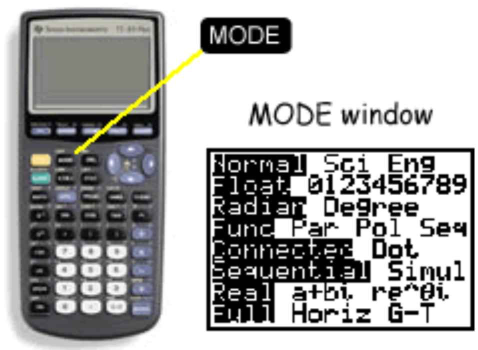


Angle	0	$\frac{\pi}{6}$, or $30°$	$\frac{\pi}{4}$, or $45°$	$\frac{\pi}{3}$, or $60°$	$\frac{\pi}{2}$, or $90°$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1





Using a Calculator to Find Sine and Cosine

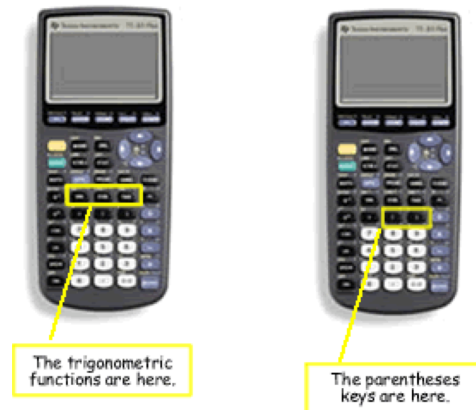


To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate cos(30)on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

HOW TO

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

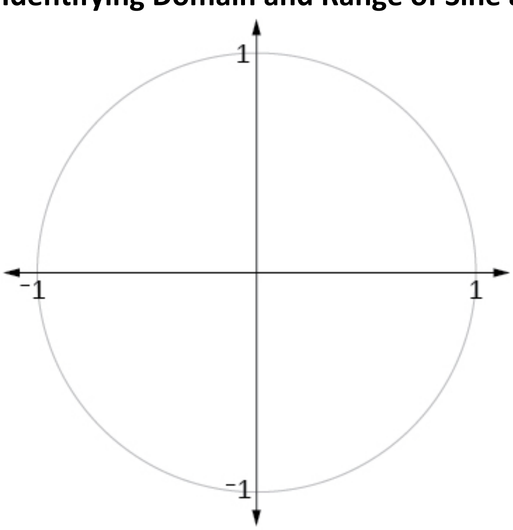


Examples

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Evaluate $\sin\left(\frac{\pi}{3}\right)$

Identifying Domain and Range of Sine and Cosine Functions



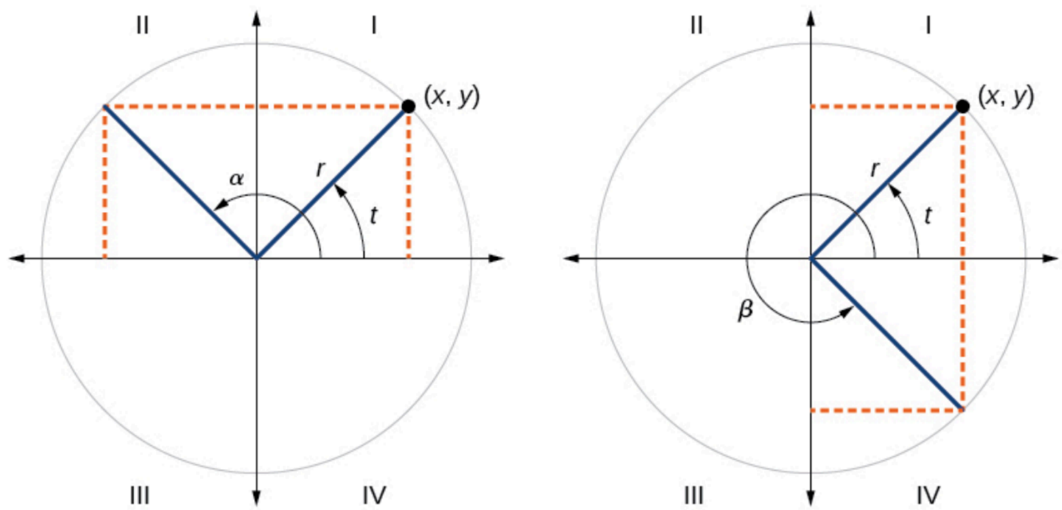
What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be ________ ____________ ____________________.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in Figure. The bounds of the x-coordinate are[____, ____].The bounds of the y-coordinate are also[____,______].Therefore, the range of both the sine and cosine functions is[−1,1].

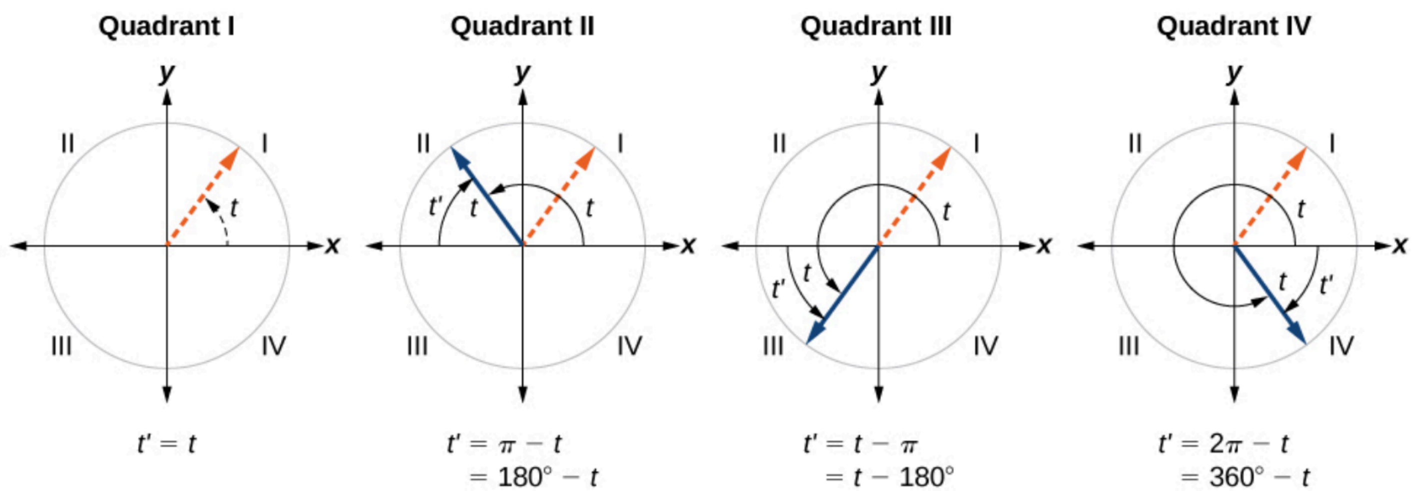
Finding Reference Angles

For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the ___-coordinate on the unit circle, the other angle with the same ______ will share the same *y*-value, but have the ____________________ *x*-value. Therefore, its _______ value will be the opposite of the first angle's cosine value. Likewise, there will be an angle in the fourth quadrant with the _________ cosine as the original angle. The angle with the same cosine will share the _________ *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$



Recall that an angle's reference angle is the acute angle, *t*, formed by the terminal side of the angle *t* and the horizontal axis. A reference angle is always an angle between 0 and 90°,or 0 and $\frac{\pi}{2}$ radians.



HOW TO

Given an angle between 0 and 2π,find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π,add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Examples

Find the reference angle of $225°$

Find the reference angle of $\frac{5\pi}{3}$

Using Reference Angles to Evaluate Trigonometric Functions

A GENERAL NOTE: USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

HOW TO

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the *x*-values in the quadrant of the original angle.
4. Give the sine the same sign as the *y*-values in the quadrant of the original angle.

HOW TO

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the *x*-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Examples

1. Use the reference angle of $315°$ to find $\cos(315°)$ and $\sin(315°)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$