Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

Basic Properties of Logarithms

- For any base a $(0<a$ and $a\neq 1)$
- $\log_a 1 = 0$
  - $a^0 = 1$
- $\log_a a = 1$
  - $a^1 = a$
- $\log_a a^x = x$
  - $a^x = a^x$
- $a^{\log_a x} = x$
  - $a^x = \log_a x$

Recall that we use the *product rule of exponents* to combine the product of exponents by adding:

$$x^a x^b = x^{a+b}$$

We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\log_b(MN) = \log_b(b^m b^n)$$

A GENERAL NOTE: THE PRODUCT RULE FOR LOGARITHMS

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

HOW TO

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1.1. Factor the argument completely, expressing each whole number factor as a product of primes.

22. Write the equivalent expression by summing the logarithms of each factor.

Examples

Expand $\log_3(30x(3x+4))$.

Expand $\log_b(8k)$.

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting:

$$x^{a/b}=x^{a-b}.$$

The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N).$$

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m=M$ and $b^n=N$. It follows that

$$\log_b\left(\tfrac{M}{N}\right) = \log_b\left(\tfrac{b^m}{b^n}\right)$$

A GENERAL NOTE: THE QUOTIENT RULE FOR LOGARITHMS

The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right)=\log_b M-\log_b N$$

HOW TO

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1.1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2.2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3.3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Examples

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$

Using the Power Rule for Logarithms

A GENERAL NOTE: THE POWER RULE FOR LOGARITHMS

The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b\left(M^n\right) = n\log_b M$$

$$\log_b\left(x^2\right) = \log_b(x \cdot x)$$

HOW TO

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1.1. Express the argument as a power, if needed.

22. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Examples

Expand $\log_2 x^5$.

Expand $\ln x^2$.

Expand $\log_3 (25)$ using the power rule for logs.

Expand $\ln\left(\frac{1}{x^2}\right)$

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Examples

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Expand $\log\left(\frac{x^2 y^3}{z^4}\right)$

Expand $\log\left(\sqrt{x}\right)$

Expand $\ln\left(\sqrt[3]{x^2}\right)$

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$

Condensing Logarithmic Expressions

HOW TO

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1.1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.

22. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.

33. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Examples

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Condense $\log 3 - \log 4 + \log 5 - \log 6$.

Condense $\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right)$

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Using the Change-Of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the change-of-base formula to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\log_n(b^y) = \log_n M$$

A GENERAL NOTE: THE CHANGE-OF-BASE FORMULA

The **change-of-base formula** can be used to evaluate a logarithm with any base.

For any positive real numbers $M, b,$ and $n,$ where $n \neq 1$ and $b \neq 1,$

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

HOW TO

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1.1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.

22. Rewrite the log as a quotient using the change-of-base formula
    - The numerator of the quotient will be a logarithm with base n and argument M.
    - The denominator of the quotient will be a logarithm with base n and argument b.

Examples

Change $\log_5 3$ to a quotient of natural logarithms.

Change $\log_{0.5} 8$ to a quotient of natural logarithms.

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Evaluate $\log_5(100)$ using the change-of-base formula.