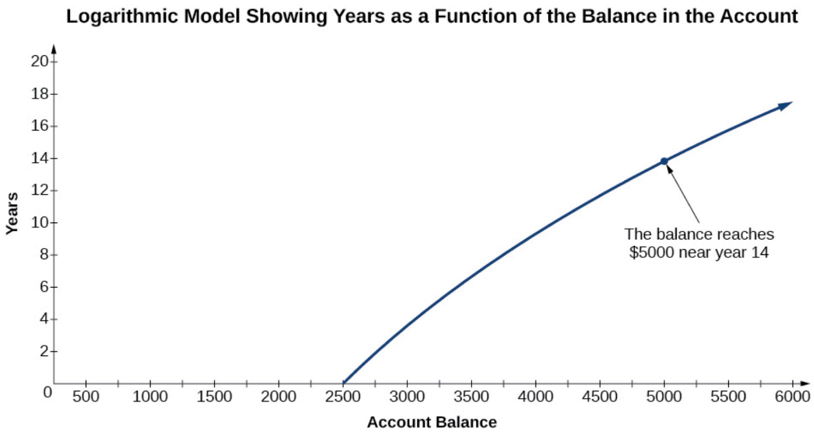
6.4 – Graphs of Logarithmic Functions

How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest$2500in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year *t* can be found with the equation $A=2500e^{0.05t}$.



But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure shows this point on the logarithmic graph.

Finding the Domain of a Logarithmic Function

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0, b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y=\log_b(x)$behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

HOW TO

Given a logarithmic function, identify the domain.

1.1. Set up an inequality showing the argument greater than zero.
22. Solve for x.
33. Write the domain in interval notation.

Example

What is the domain of $f(x) = \log_2(x+3)$'

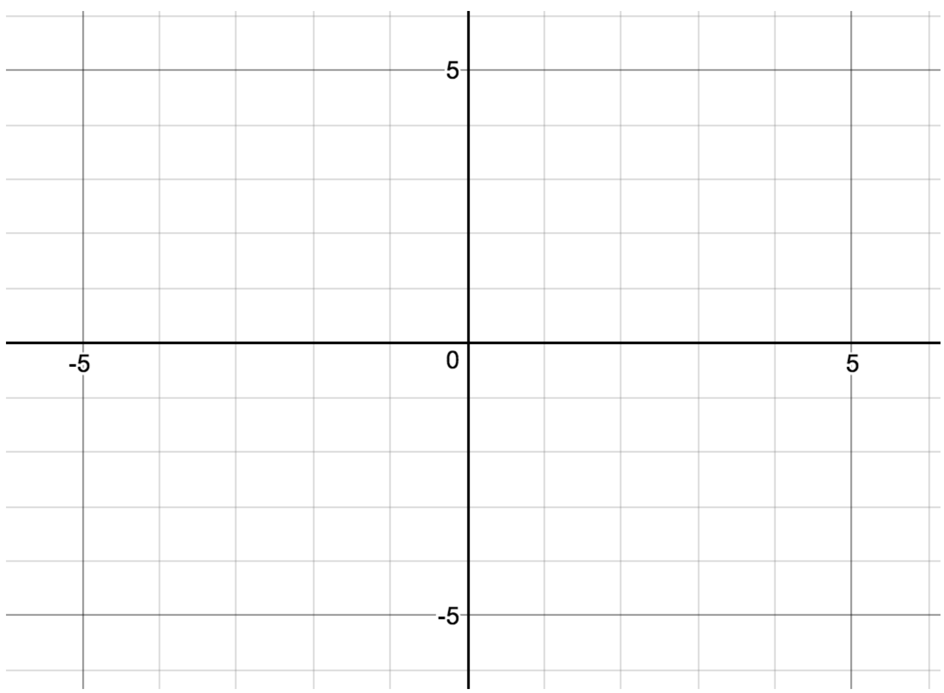
What is the domain of $f(x) = \log_5(x-2)+1$?

Graphing Logarithmic Function

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table.

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$



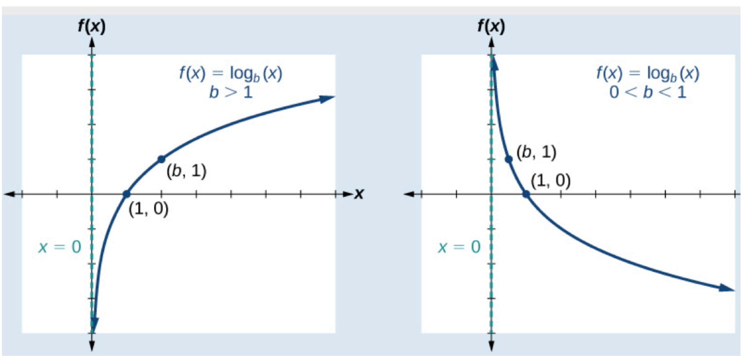
Observe the following from the graph:

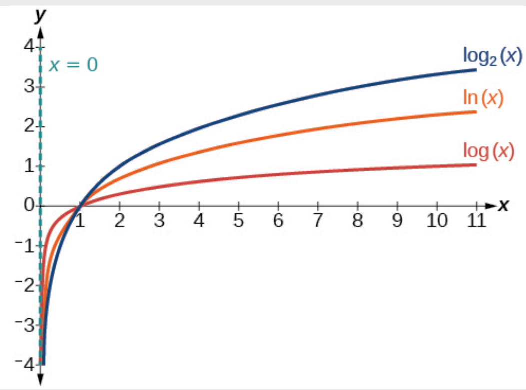
- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

A GENERAL NOTE: CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $F(X) = \text{LOG}_B(X)$

For any real number x and constant $b > 0, b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- *x*-intercept: $(1, 0)$ and key point $(b, 1)$
- *y*-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$





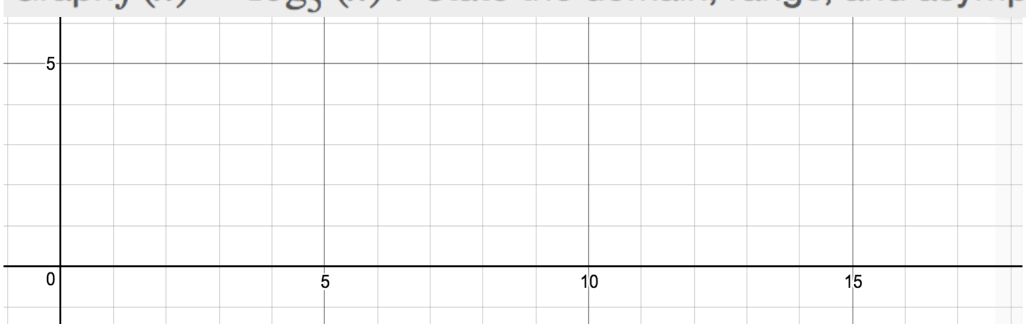
HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

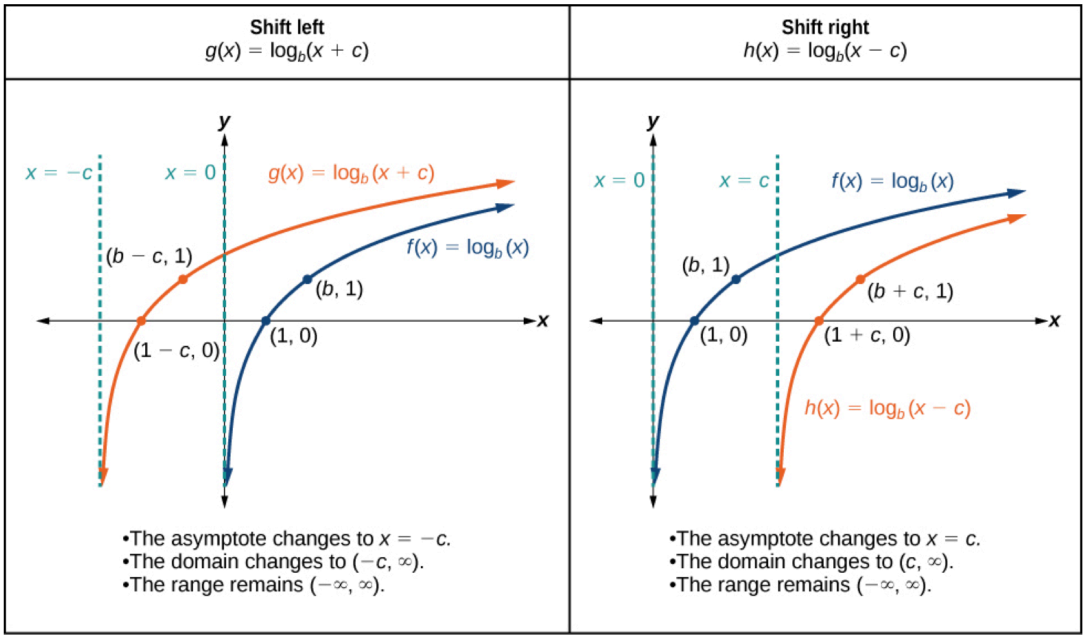
11. Draw and label the vertical asymptote, $x = 0$.
22. Plot the *x*-intercept, $(1, 0)$.
33. Plot the key point $(b, 1)$.
44. Draw a smooth curve through the points.
55. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Example

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.



Graphing Transformations of Logarithmic Functions
Graphing a Horizontal Shift



A GENERAL NOTE: HORIZONTAL SHIFTS OF THE PARENT FUNCTION $Y = \text{LOG}_B(X)$

For any constant c, the function $f(x) = \log_b(x+c)$
- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

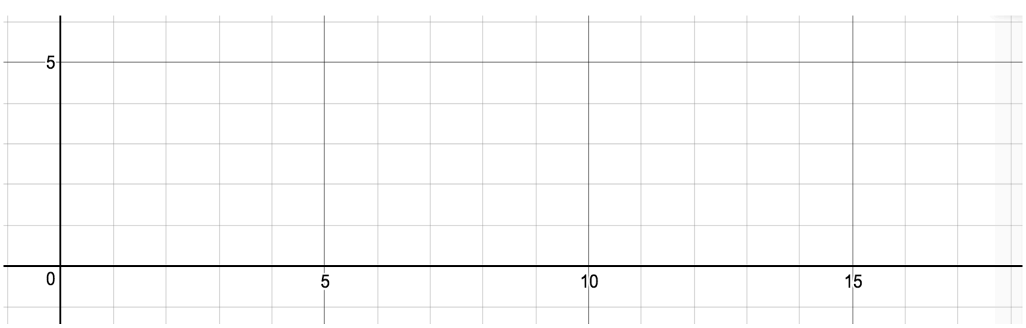
HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x+c)$, graph the translation.

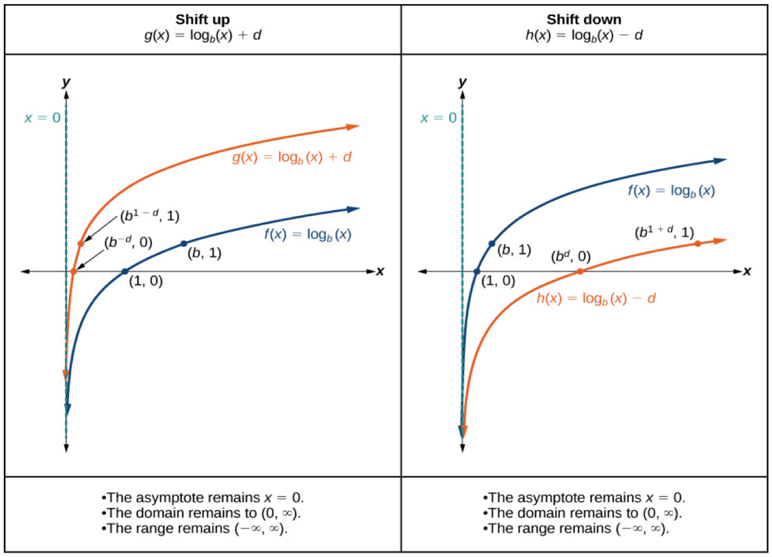
1. 1. Identify the horizontal shift:
   1a. If $c > 0$, shift the graph of $f(x) = \log_b(x)$ left c units.
   2b. If $c < 0$, shift the graph of $f(x) = \log_b(x)$ right c units.
2. 2. Draw the vertical asymptote $x = -c$.
3. 3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. 4. Label the three points.
5. 5. The Domain is $(-c, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.



Graphing a Vertical Shift



A GENERAL NOTE: VERTICAL SHIFTS OF THE PARENT FUNCTION $Y = \text{LOG}_B(X)$

For any constant d, the function $f(x) = \log_b(x) + d$
- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1.1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down d units.

22. Draw the vertical asymptote $x = 0$.

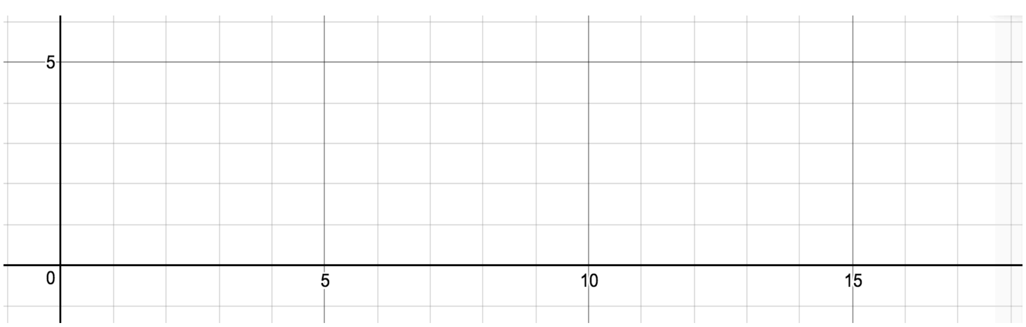
33. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.

44. Label the three points.

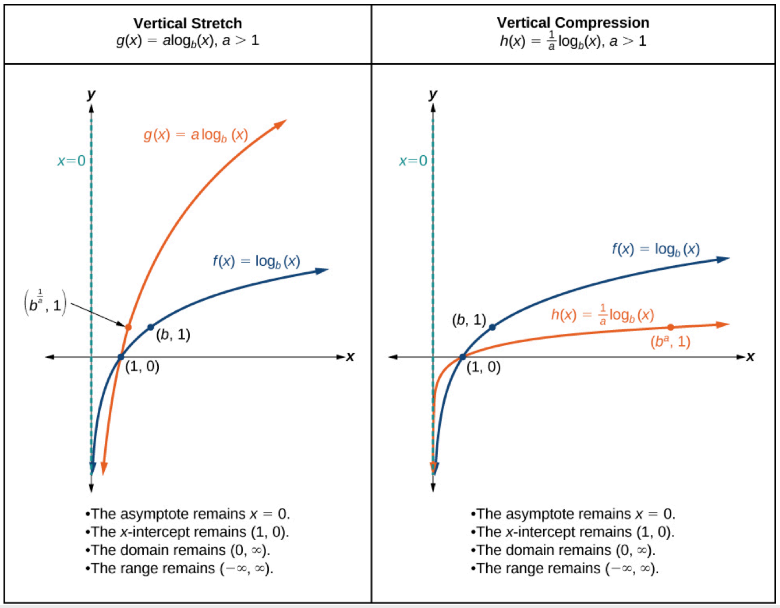
55. The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x = 0$.

Example

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.



Graphing Stretches and Compressions



A GENERAL NOTE: VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $a > 1$, the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

HOW TO

Given a logarithmic function with the form $f(x) = a\log_b(x)$, $a > 0$, graph the translation.

1.1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
   - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

22. Draw the vertical asymptote $x = 0$.

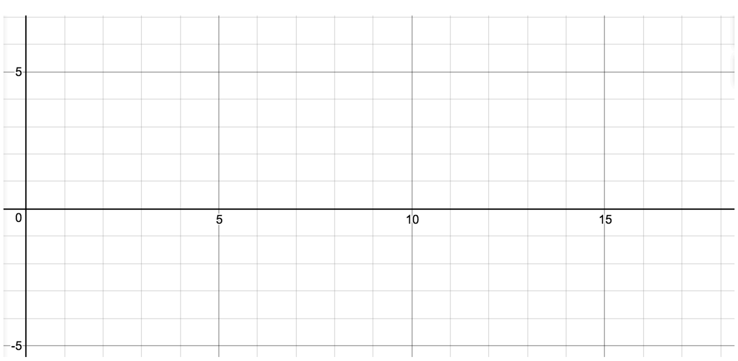
33. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.

44. Label the three points.

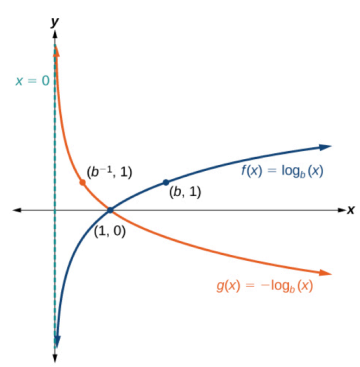
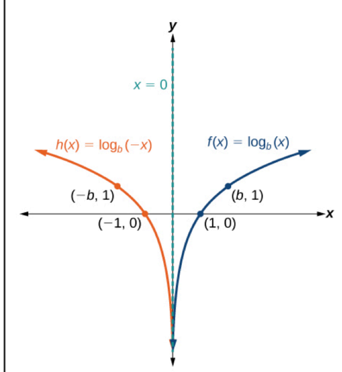
55. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.



Graphing Reflections

Reflection about the *x*-axis $g(x) = \log_b(x)$, $b > 1$	Reflection about the *y*-axis $h(x) = \log_b(-x)$, $b > 1$
	
•The reflected function is decreasing as x moves from zero to infinity. •The asymptote remains $x = 0$. •The x-intercept remains $(1, 0)$. •The key point changes to $(b^{-1}, 1)$ •The domain remains $(0, \infty)$. •The range remains $(-\infty, \infty)$.	•The reflected function is decreasing as x moves from negative infinity to zero. •The asymptote remains $x = 0$. •The x-intercept changes to $(-1, 0)$. •The key point changes to $(-b, 1)$ •The domain changes to $(-\infty, 0)$. •The range remains $(-\infty, \infty)$.

A GENERAL NOTE: REFLECTIONS OF THE PARENT FUNCTION $Y = \text{LOG}_B(X)$

The function $f(x) = -\log_b(x)$
- reflects the parent function $y = \log_b(x)$ about the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$
- reflects the parent function $y = \log_b(x)$ about the y-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

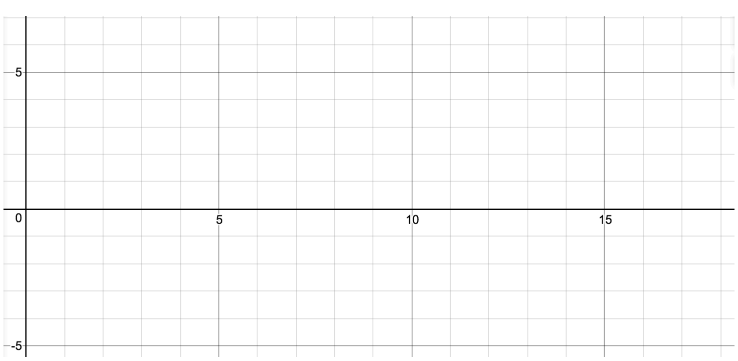
HOW TO

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1.1. Draw the vertical asymptote, $x = 0$.	1.1. Draw the vertical asymptote, $x = 0$.
23.. Plot the x-intercept, $(1, 0)$.	23.. Plot the x-intercept, $(1, 0)$.
34.. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the x-axis.	34.. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the y-axis.
45.. Draw a smooth curve through the points.	45.. Draw a smooth curve through the points.
56.. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	56.. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Example

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.



HOW TO

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1.1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
22. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
33. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x, for the point(s) of intersection.

Examples

Solve $4\ln(x)+1=-2\ln(x-1)$ graphically. Round to the nearest thousandth.

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

Summary of Translations of the Logarithmic Functions

Translations of the Parent Function $y=\log_b(x)$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$y=\log_b(x+c)+d$
Stretch and Compress • Stretch if $\lvert a\rvert>1$ • Compression if $\lvert a\rvert<1$	$y=a\log_b(x)$
Reflect about the x-axis	$y=-\log_b(x)$
Reflect about the y-axis	$y=\log_b(-x)$
General equation for all translations	$y=a\log_b(x+c)+d$

A GENERAL NOTE: TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the parent logarithmic function, $y=\log_b(x)$, have the form

$$f(x)=a\log_b(x+c)+d$$

where the parent function, $y=\log_b(x)$, $b>1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $\lvert a\rvert$ if $\lvert a\rvert>0$.
- compressed vertically by a factor of $\lvert a\rvert$ if $0<\lvert a\rvert<1$.
- reflected about the x-axis when $a<0$.

For $f(x)=\log(-x)$, the graph of the parent function is reflected about the y-axis.