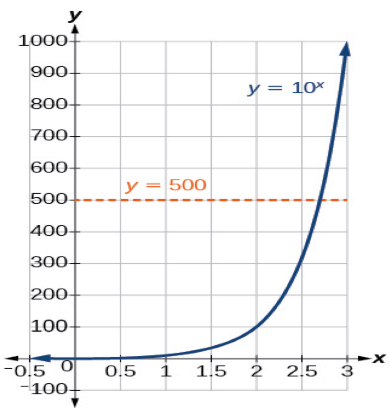
6.3 – Logarithmic Functions

Converting from Logarithmic to Exponential

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4}=10^4=10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is$10^x=500$, where x represents the difference in magnitudes on the ________________________ Scale.

How would we solve for x?

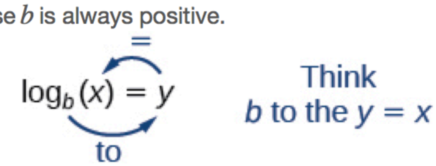


Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure passes the horizontal line test. The exponential function $y=b^x$ is _______-to-________, so its inverse, $x=b^y$ is also a ________________. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y=\log_b(x)$.The base b ________________________________ of a number is the exponent by which we must raise b to get that number.

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



A GENERAL NOTE: DEFINITION OF THE LOGARITHMIC FUNCTION

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, "the logarithm with base b of x" or the "log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q&A

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

HOW TO

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1.1. Examine the equation $y = \log_b x$ and identify $b, y, \text{and} x$.

22. Rewrite $\log_b x = y$ as $b^y = x$.

Examples

Write the following logarithmic equations in exponential form.

1a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

2b. $\log_3(9) = 2$

Write the following logarithmic equations in exponential form.

1a. $\log_{10}(1{,}000{,}000) = 6$

2b. $\log_5(25) = 2$

Write the following exponential equations in logarithmic form.

1a. $2^3 = 8$

2b. $5^2 = 25$

3c. $10^{-4} = \frac{1}{10{,}000}$

Write the following exponential equations in logarithmic form.

1a. $3^2 = 9$

2b. $5^3 = 125$

3c. $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

HOW TO

Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally.

1.1. Rewrite the argument x as a power of b : $b^y = x$.

22. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x?"

Examples

Solve $y = \log_4(64)$ without using a calculator.

Solve $y = \log_{121}(11)$ without using a calculator.

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithm

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression log(x) means $\log_{10}(x)$.

We call a base-10 logarithm a ________________________________ **logarithm**.

Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

A GENERAL NOTE: DEFINITION OF THE COMMON LOGARITHM

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

HOW TO

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1.1. Rewrite the argument x as a power of 10 : $10^y = x$.

22. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Examples

Evaluate $y = \log(1000)$ without using a calculator.

Evaluate $y = \log(1{,}000{,}000)$.

HOW TO

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1.1. Press **[LOG]**.

22. Enter the value given for x, followed by **[ )]**.

33. Press **[ENTER]**.

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

A GENERAL NOTE: DEFINITION OF THE NATURAL LOGARITHM

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

HOW TO

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1.1. Press **[LN]**.
22. Enter the value given for x, followed by **[ )]**.
33. Press **[ENTER]**.

Examples

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Evaluate $\ln(-500)$.

Extra Practice

$\log_3\left(\frac{1}{27}\right)$

$\log_6(\sqrt{6})$

$\log_2\left(\frac{1}{8}\right)+4$

$6\log_8(4)$

$\log(0.001)$

$\log(1)+7$

$2\log(100^{-3})$

$25\ln(e^{\frac{2}{5}})$

The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI=\log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?