6.1 – Exponential Functions

Identifying Exponential Functions

For us to gain a clear understanding of how an ________________________ function behaves, let us contrast exponential growth with linear growth, which we are already familiar with. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1.

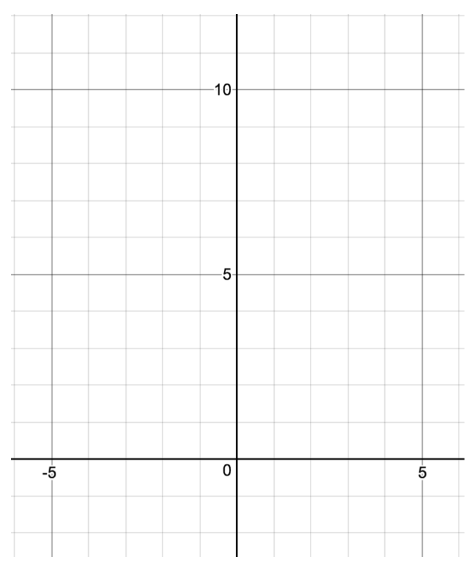
x	$f(x) = 2^x$	$g(x) = 2x$

- ______________________________ **growth** refers to the original value from the range increases by the same ________________________ over **EQUAL** increments found in the domain.
- ______________________________ **growth** refers to the original value from the range increases by the same _____________________ over equal increments found in the domain.

Apparently, the difference between "**the same percentage**" and "**the same amount**" is quite significant. For exponential growth, over equal increments, the constant ______________________________ rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant ______________________________ rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

Let's take a closer look at the function $f(x) = 2^x$ from our example.

x	f(x)
-3	
-2	
-1	
0	
1	
2	
3	



Let's define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty, f(x) \to \infty$,
- as $x \to -\infty, f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the y-intercept is 1.

A GENERAL NOTE: EXPONENTIAL FUNCTION

For any real number x, an exponential function is a function with the form

$$f(x) = ab^x$$

where

- a is the a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$, and the horizontal asymptote is $y = 0$.

Examples

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Defining Exponential Growth

A GENERAL NOTE: EXPONENTIAL GROWTH

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

Examples

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example?

Finding Equations of Exponential Functions

HOW TO

Given two data points, write an exponential model.

1.1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.

22. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.

33. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Examples

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

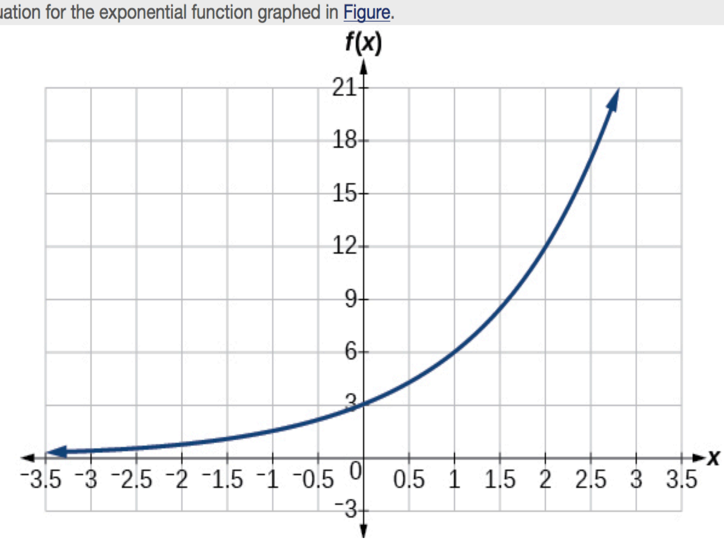
HOW TO

Given the graph of an exponential function, write its equation.

11. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
22. If one of the data points is the *y*-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
33. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
44. Write the exponential function, $f(x) = a(b)^x$.

Example

Find an equation for the exponential function graphed in Figure.



HOW TO

Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

11. Press **[STAT].**
22. Clear any existing entries in columns **L1** or **L2.**
33. In **L1**, enter the *x*-coordinates given.
44. In **L2**, enter the corresponding *y*-coordinates.
55. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
66. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

Example

Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Applying the Compound-Interest Formula
Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use ________________________________ interest. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

A GENERAL NOTE: THE COMPOUND INTEREST FORMULA

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Compounding Periods					
Annually	Semiannually	Quarterly	Monthly	Weekly	Daily
n=1	n=2	n=4	n=12	n=52	n=360

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Evaluating Functions with Base *e*

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed below

Frequency	$A(t) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	$2
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1 + \frac{1}{8766}\right)^{8766}$	$2.718127
Once per minute	$\left(1 + \frac{1}{525960}\right)^{525960}$	$2.718279
Once per second	$\left(1 + \frac{1}{31557600}\right)^{31557600}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

A GENERAL NOTE: THE NUMBER *E*

The letter *e* represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n, \text{as } n \text{ increases without bound}$$

The letter *e* is used as a base for many real-world exponential models. To work with base *e*, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Examples

Calculate $e^{3.14}$. Round to five decimal places.

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

A GENERAL NOTE: THE CONTINUOUS GROWTH/DECAY FORMULA

For all real numbers t,and all positive numbers a and r,continuous growth or decay is represented by the formula

$$A(t) = ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

HOW TO

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   - 1a. If the problem refers to continuous growth, then $r > 0$.
   - 2b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?