6.7 – Exponential and Logarithmic Models

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y=A_0e^{kt}$$

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape. It is important to remember that, although parts of each of the two graphs seem to lie on the *x*-axis, they are really a tiny distance above the *x*-axis.

A GENERAL NOTE: CHARACTERISTICS OF THE EXPONENTIAL FUNCTION, $Y = A_0E^{KT}$

An exponential function with the form $y = A_0e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty,\ \infty)$
- range: $(0, \infty)$
- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure)
- decreasing if $k < 0$ (see Figure)

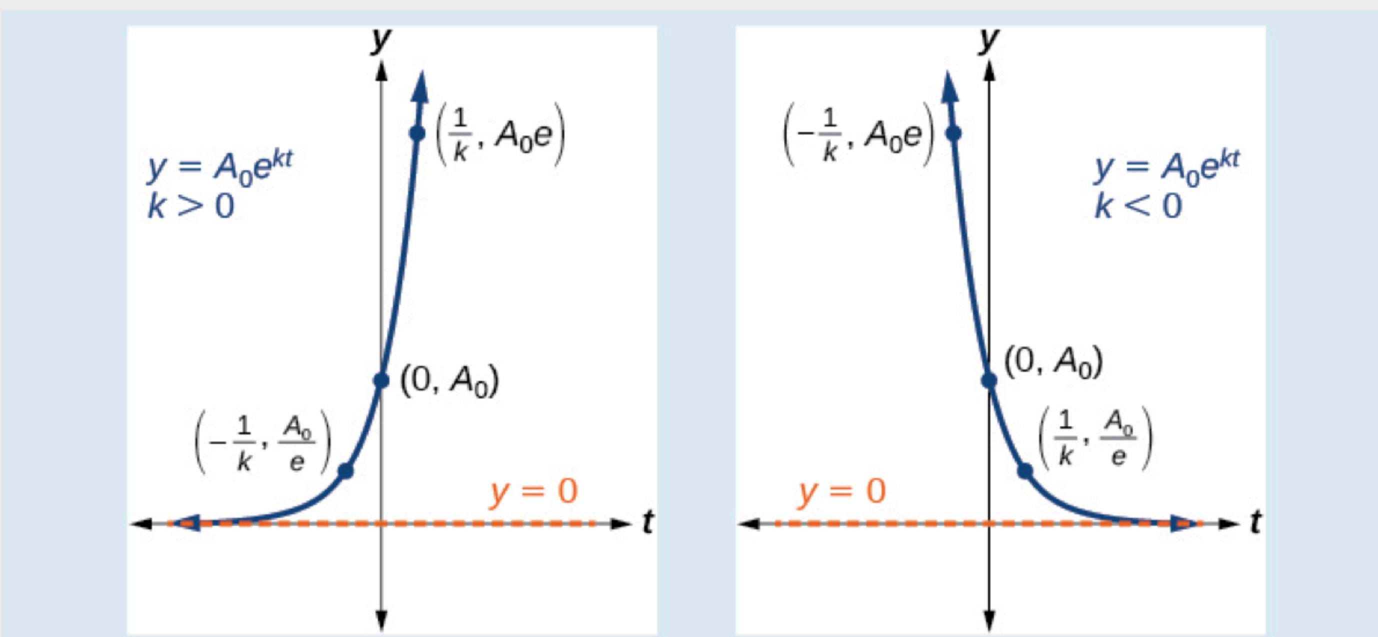


Figure 4. An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

Example

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Half-Life

We now turn to exponential decay. One of the common terms associated with exponential _______________, as stated above, is **__________-__________**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay. To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_o e^{kt}$$

HOW TO

Given the half-life, find the decay rate.

1.1. Write $A = A_o e^{kt}$.

22. Replace A by $\frac{1}{2}A_0$ and replace t by the given half-life.

33. Solve to find k. Express k as an exact value (do not round).

Note: *It is also possible to find the decay rate using* $k = -\frac{\ln(2)}{t}$.

Example

The half-life of plutonium-244 is 80,000,000 years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

Radiocarbon Dating

The formula for radioactive decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, the formula for the amount of carbon-14 remaining after *t* years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

- A is the amount of carbon-14 remaining
- A_0 is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$A = A_0 e^{kt}$$

HOW TO

Given the percentage of carbon-14 in an object, determine its age.

1.1. Express the given percentage of carbon-14 as an equivalent decimal, k.

22. Substitute for k in the equation $t = \dfrac{\ln(r)}{-0.000121}$ and solve for the age, t.

Examples

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the ______________________________________ time.

The formula is derived as follows:

$$2A_0 = A_0 e^{kt}$$

Examples

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Using Newton's Law of Cooling

A GENERAL NOTE: NEWTON'S LAW OF COOLING

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

HOW TO

Given a set of conditions, apply Newton's Law of Cooling.

1.1. Set T_s equal to the y-coordinate of the horizontal asymptote (usually the ambient temperature).
22. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters A and k.
33. Substitute in the desired time to find the temperature or the desired temperature to find the time.

Example

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

The ______________________________ growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the ________________________ capacity. For constants *a, b,* and *c,* the logistic growth of a population over time *x* is represented by the model

A GENERAL NOTE: LOGISTIC GROWTH

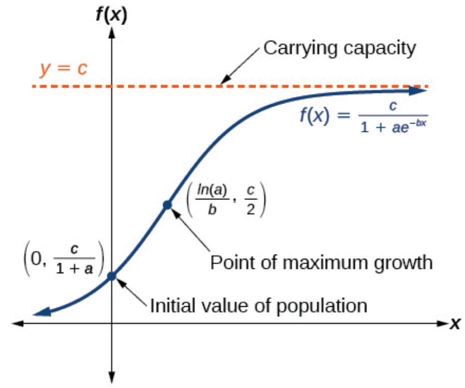
The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

$$f(x) = \frac{c}{1 + ae^{-bx}}$$



Example

Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.