5.3 – Polynomial Functions

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

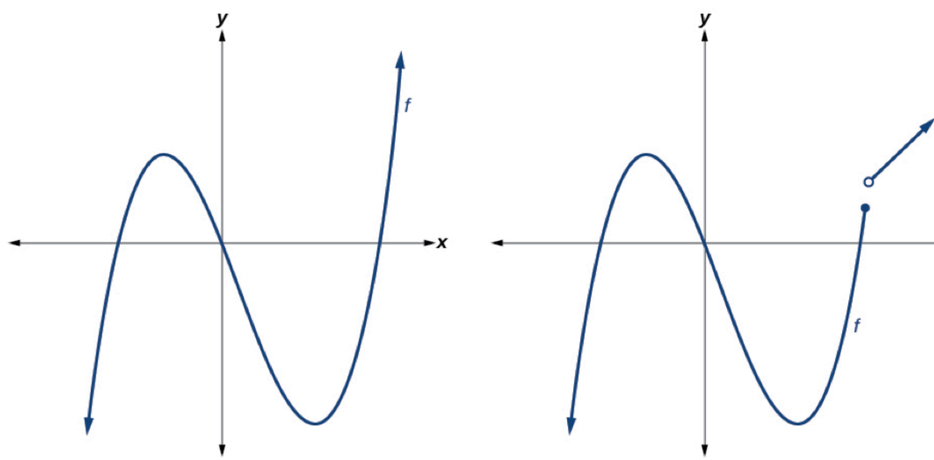
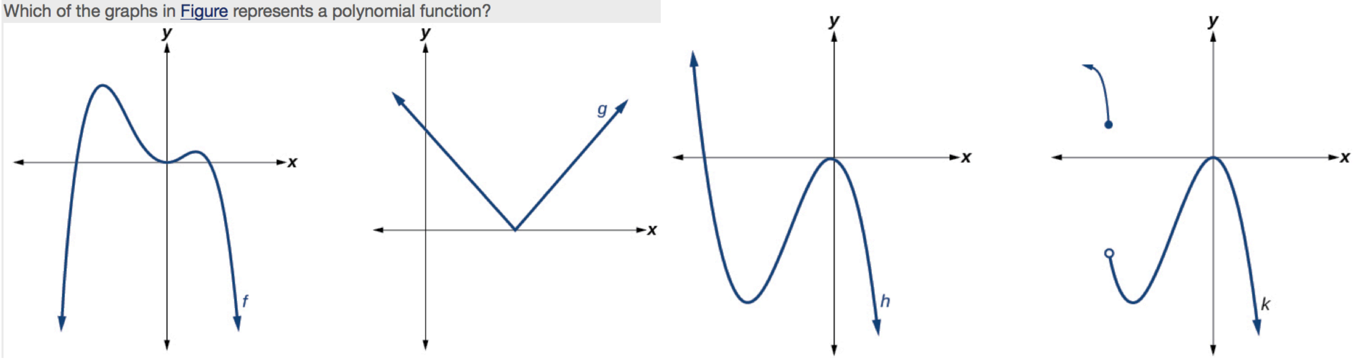


Figure 1.

Examples

Which of the graphs in Figure represents a polynomial function?



Q&A

Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x)=0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros. We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

HOW TO

Given a polynomial function f, find the x-intercepts by factoring.

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Examples

Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the *y*- and *x*-intercepts of $g(x) = (x - 2)^2(2x + 3)$.

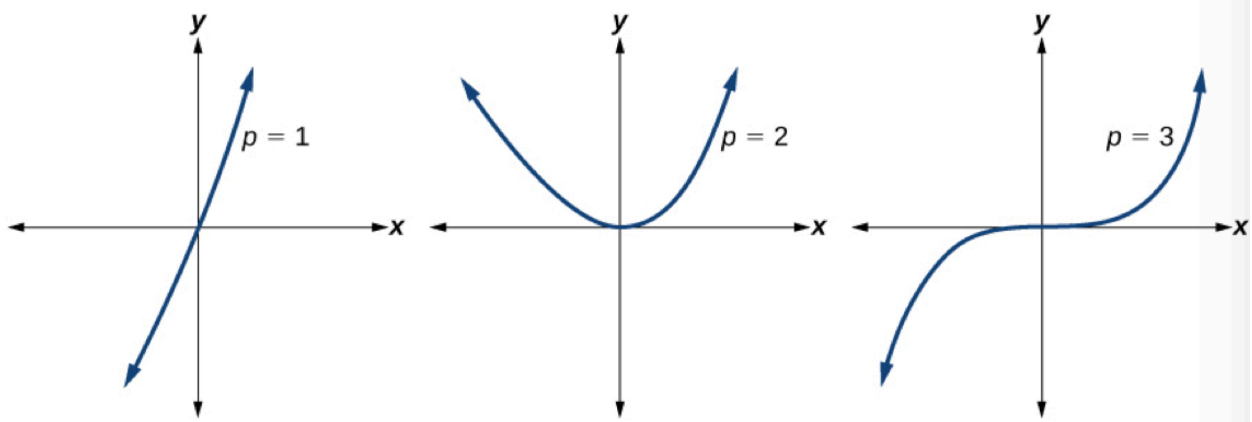
Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Find the *y*- and *x*-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Identifying Zeros and Their Multiplicities

The number of times a given factor appears in the factored form of the equation of a polynomial is called the ____. For zeros with even multiplicities, the graphs *touch* or are tangent to the *x*-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the *x*-axis. See [Figure] for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.



Single zero

Zero with multiplicity 2

Zero with multiplicity 3

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the *x*-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the *x*-axis.

GRAPHICAL BEHAVIOR OF POLYNOMIALS AT X-INTERCEPTS

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x=h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the *x*-axis at zeros with even multiplicities. The graph will cross the *x*-axis at zeros with odd multiplicities.

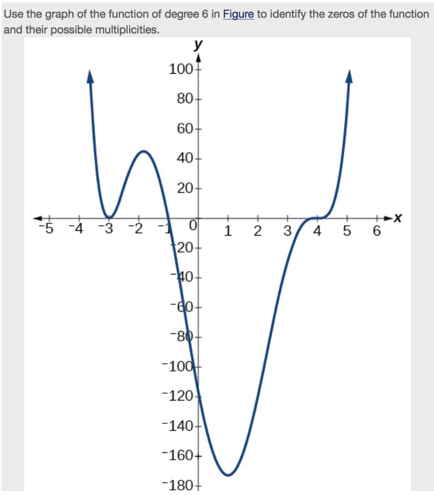
The sum of the multiplicities is the degree of the polynomial function.

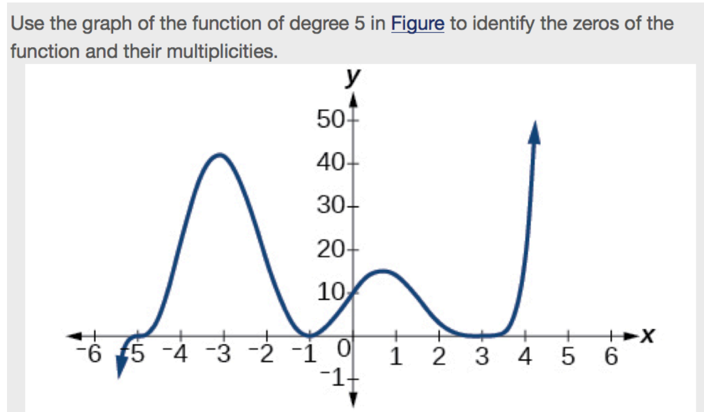
HOW TO

Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the *x*-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the *x*-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the *x*-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Examples



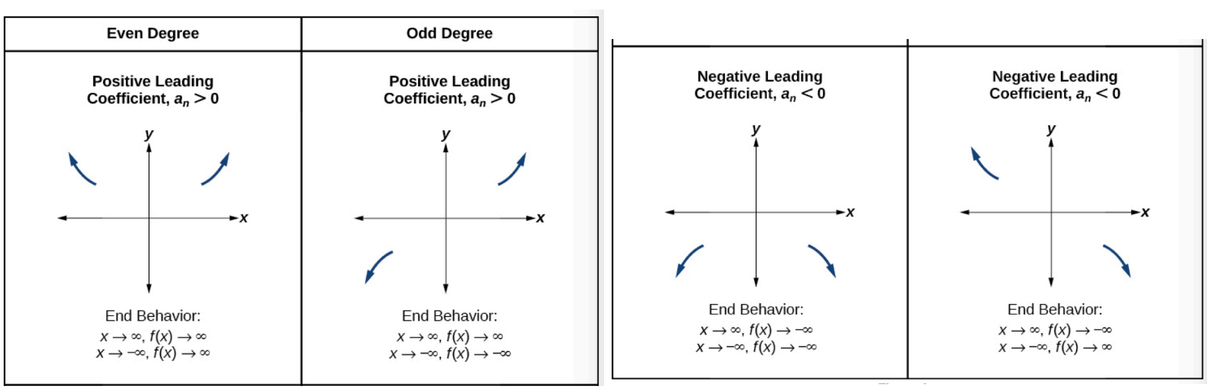


Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

will either ultimately rise or fall as *x* increases without bound and will either rise or fall as *x* decreases without bound.



Understanding the Relationship Between Degree and Turning Points

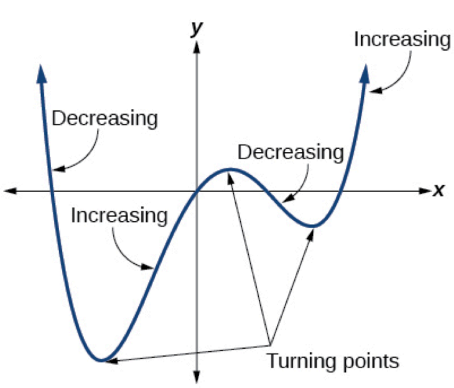


Figure 9.

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ i Figure. The graph has three turning points. This function *f* is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

A GENERAL NOTE: INTERPRETING TURNING POINTS

A **turning point** is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

A polynomial of degree n will have at most $n - 1$ turning points.

Examples

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

b. $f(x) = -(x - 1)^2 \left(1 + 2x^2\right)$

Graphing Polynomial Functions

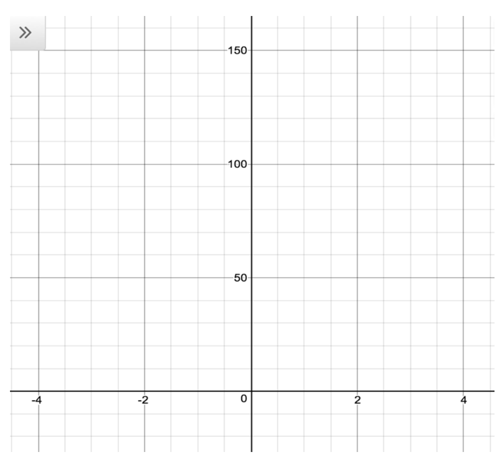
HOW TO

Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

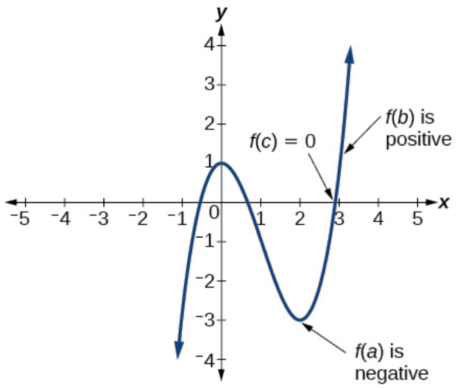
Examples

Sketch a graph of $f(x) = -2(x + 3)^2(x - 5)$.



Using the Intermediate Value Theorem

The Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure shows that there is a zero between a and b.



INTERMEDIATE VALUE THEOREM

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b fo which $f(c) = 0$.

Examples

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Writing Formulas for Polynomial Functions

FACTORED FORM OF POLYNOMIALS

If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \dots, x_n,$ then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the x-intercept.

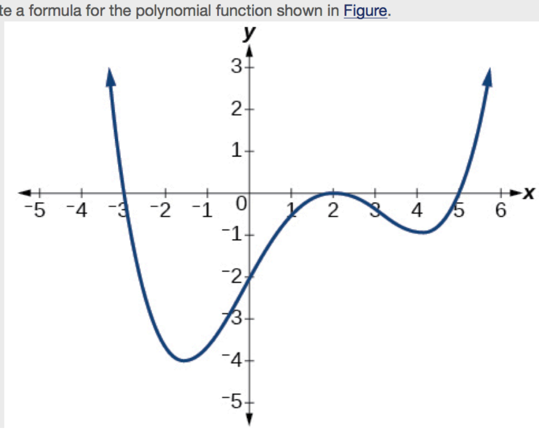
HOW TO

Given a graph of a polynomial function, write a formula for the function.

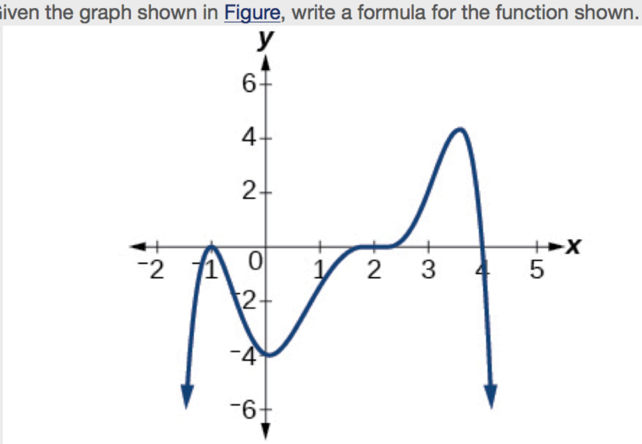
1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Examples

Write a formula for the polynomial function shown in Figure.



Given the graph shown in Figure, write a formula for the function shown.



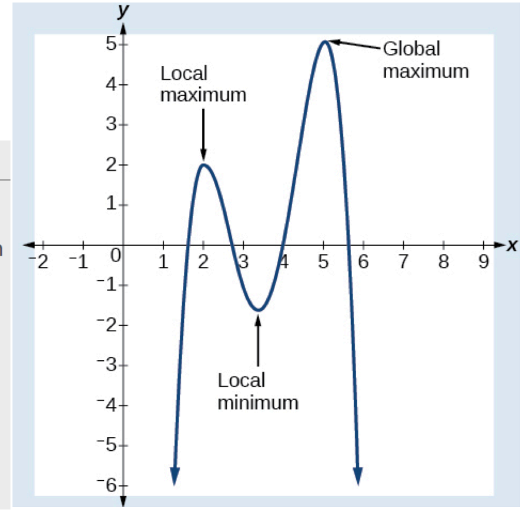
Using Local and Global Extrema

LOCAL AND GLOBAL EXTREMA

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$. If a function has a local maximum at a, then $f(a) \geq f(x)$ for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$ for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure.



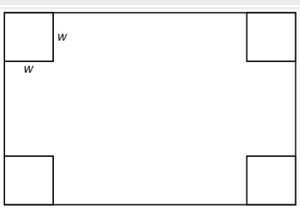
Q&A

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Examples

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic and then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.



Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.