## 2.4 – Complex Numbers

Keep in mind that the study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. In this section, we will explore a set of numbers that fills voids in the set of real numbers and find out how to work within it.

### Expressing Square Roots of Negative Numbers as Multiples of *i*

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number**.** The imaginary number $i$ is defined as the square root of $-1$.

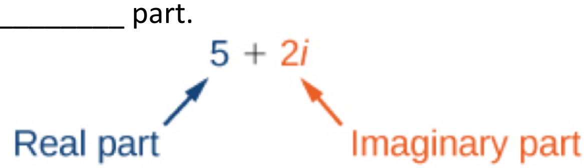
$$\sqrt{-1} = i$$

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

**Example**

$$\begin{aligned} \sqrt{-49} &= \sqrt{49 \cdot (-1)} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i \end{aligned}$$

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a+bi$ where $a$ is the ______________________ part and $b$ is the ____________________________________ part.




### A GENERAL NOTE: IMAGINARY AND COMPLEX NUMBERS

A **complex number** is a number of the form $a + bi$ where

- $a$ is the real part of the complex number.
- $b$ is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and $b$ is not equal to 0, the complex number is called a pure imaginary number. An **imaginary number** is an even root of a negative number.

### HOW TO

**Given an imaginary number, express it in the standard form of a complex number.**

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as $i$.
3. Write $\sqrt{a} \cdot i$ in simplest form.

### Examples

Express $\sqrt{-9}$ in standard form.

Express $\sqrt{-24}$ in standard form.

### Plotting a Number on the Complex Plane

A GENERAL NOTE: COMPLEX PLANE

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis, as shown in Figure.

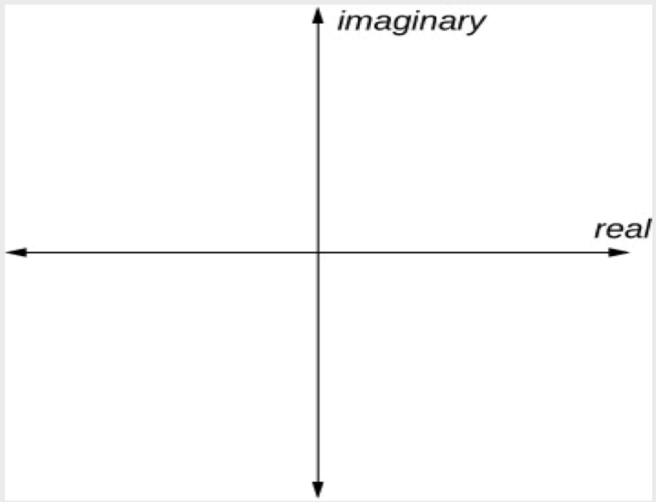



Figure 3.

HOW TO

**Given a complex number, represent its components on the complex plane.**

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

### Examples

Plot the complex number $3 - 4i$ on the complex plane.

Plot the complex number $-4 - i$ on the complex plane.

### Adding and Subtracting Complex Numbers

A GENERAL NOTE: COMPLEX NUMBERS: ADDITION AND SUBTRACTION

Adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)\,i$$

Subtracting complex numbers:

$$(a + bi) - (c + di) = (a - c) + (b - d)\,i$$

## HOW TO

**Given two complex numbers, find the sum or difference.**

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

**Examples**

Add or subtract as indicated.

1. $(3-4i)+(2+5i)$
2. $(-5+7i)-(-11+2i)$

Subtract $2+5i$ from $3-4i$.

**Multiplying Complex Numbers**

$$3(6+2i) = (3\cdot 6)+(3\cdot 2i) \quad \text{Distribute.}$$
$$= 18+6i \quad \text{Simplify.}$$

## HOW TO

**Given a complex number and a real number, multiply to find the product.**

1. Use the distributive property.
2. Simplify.

**Examples**

Find the product $4(2+5i)$.

Find the product: $\frac{1}{2}(5-2i)$.

**Multiplying Complex Numbers Together (Cycles of *i*)**

| | | |
|---|---|---|
| $i=$ | $i^5=$ | $i^9=$ |
| $i^2=$ | $i^6=$ | $i^{10}=$ |
| $i^3=$ | $i^7=$ | $i^{11}=$ |
| $i^4=$ | $i^8=$ | $i^{12}=$ |

**What is $i^{37}$?**

HOW TO

**Given two complex numbers, multiply to find the product.**

1. Use the distributive property or the FOIL method.
2. Remember that $i^2 = -1$.
3. Group together the real terms and the imaginary terms

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

**Examples**

Multiply: $(4 + 3i)(2 - 5i)$.

Multiply: $(3 - 4i)(2 + 3i)$.

$(4 - 2i)(4 + 2i)$

**What do you notice that is different about this answer?**

**Dividing Complex Numbers**

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form___________________________. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number.

$$\begin{aligned}(a+bi)(a-bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2\end{aligned}$$

## A GENERAL NOTE: THE COMPLEX CONJUGATE

The **complex conjugate** of a complex number $a + bi$ is $a - bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

**Examples**

Find the complex conjugate of each number.

1. $2 + i\sqrt{5}$
2. $-\frac{1}{2}i$

Find the complex conjugate of $-3 + 4i$.

**Given two complex numbers, divide one by the other.**

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

**Examples**

$\frac{-5+3i}{2i}$

Divide: $(2 + 5i)$ by $(4 - i)$.

$-\sqrt{-4} - 4\sqrt{-25}$

Evaluate $(1-i)^k$ for $k = 2, 6, \text{and } 10$. Predict the value if $k = 14$.

$\frac{1}{i} + \frac{4}{i^3}$

$\frac{3+2i}{2+i} + (4+3i)$