# 3.5 – Transformations of Functions

Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

### Graphing Functions Using Vertical and Horizontal Shifts

One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

### Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x)=f(x)+k$, the function $f(x)$ is shifted vertically $k$ units. See Figure for an example.

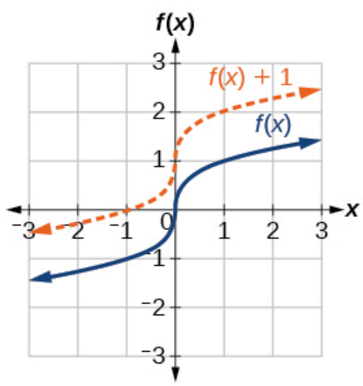



Figure 2. Vertical shift by $k=1$ of the cube root function $f(x)=\sqrt[3]{x}$.

A GENERAL NOTE: VERTICAL SHIFT

Given a function $f(x)$, a new function $g(x)=f(x)+k$, where $k$ is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by $k$ units. If $k$ is positive, the graph will shift up. If $k$ is negative, the graph will shift down.

Example

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure shows the area of open vents $V$ (in square feet) throughout the day in hours after midnight, $t$. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

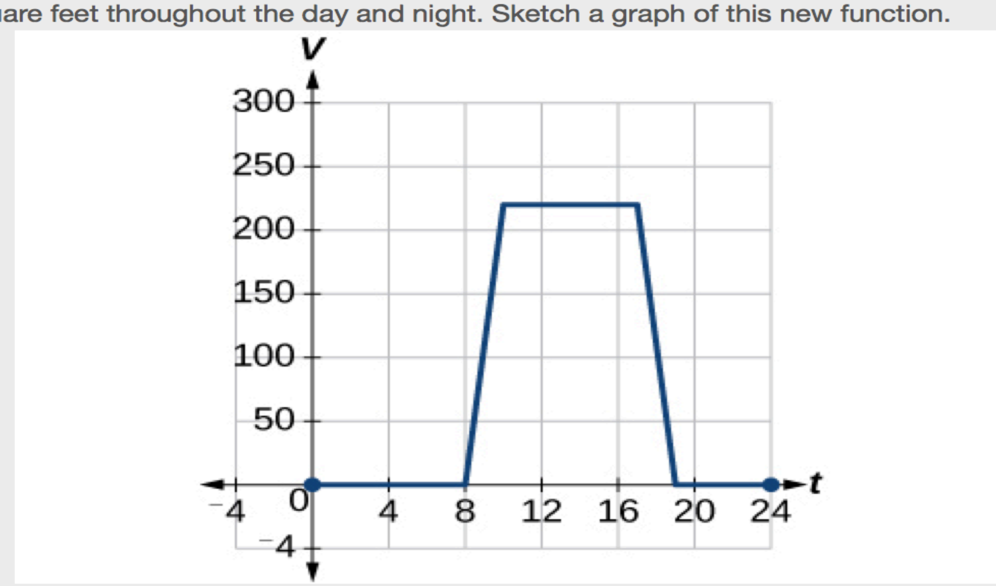



Figure 3.

## HOW TO

**Given a tabular function, create a new row to represent a vertical shift.**

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

**Example**

**a.**

A function $f(x)$ is given in Table. Create a table for the function $g(x) = f(x) - 3$.

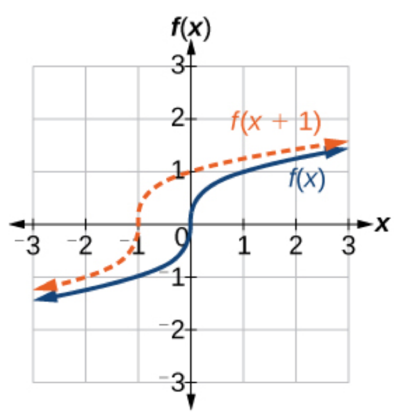
| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**b.**

The function $h(t) = -4.9t^2 + 30t$ gives the height $h$ of a ball (in meters) thrown upward from the ground after $t$ seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

**Identifying Horizontal Shifts**

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure.

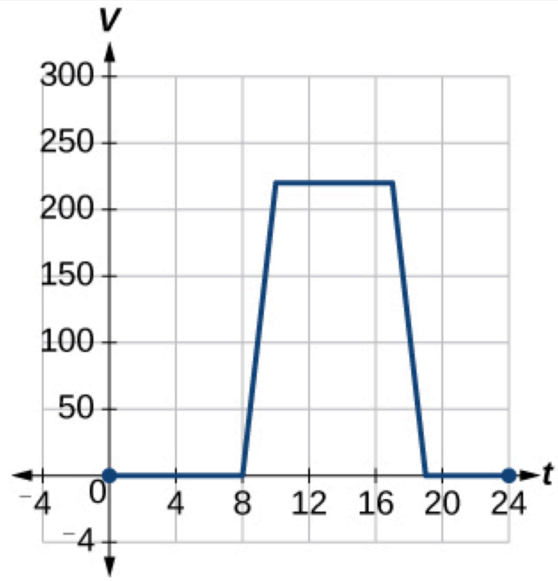


## A GENERAL NOTE: HORIZONTAL SHIFT

Given a function $f$, a new function $g(x) = f(x - h)$, where $h$ is a constant, is a **horizontal shift** of the function $f$. If $h$ is positive, the graph will shift right. If $h$ is negative, the graph will shift left.

**Example**

Returning to our building airflow example from Figure, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.




## HOW TO

**Given a tabular function, create a new row to represent a vertical shift.**

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

**Example**

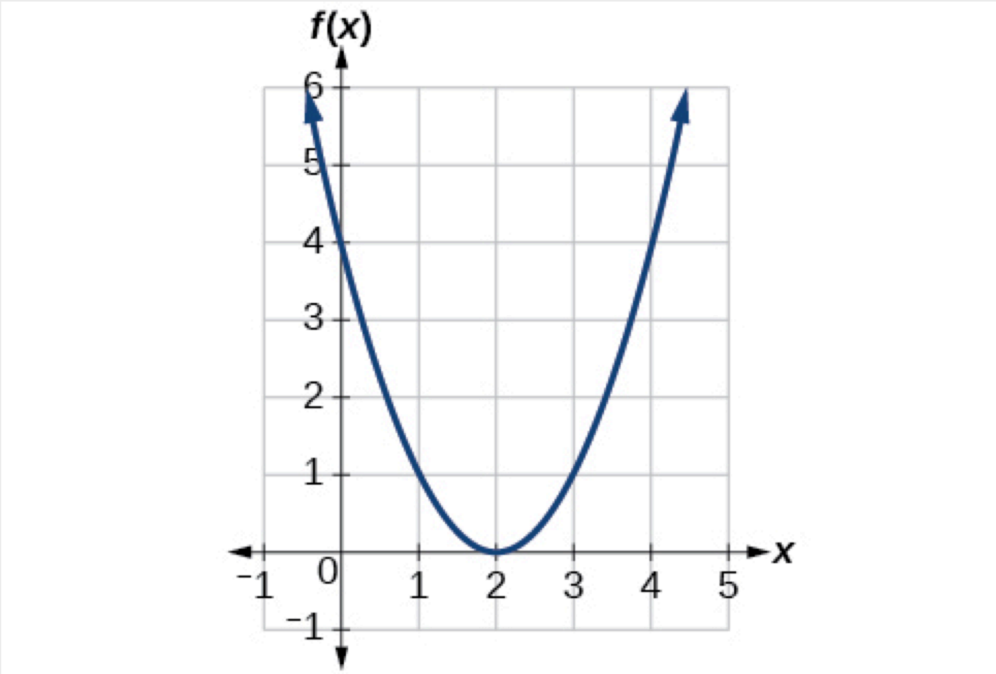
**a.**

A function $f(x)$ is given in Table. Create a table for the function $g(x) = f(x - 3)$.

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**b.**

Figure represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

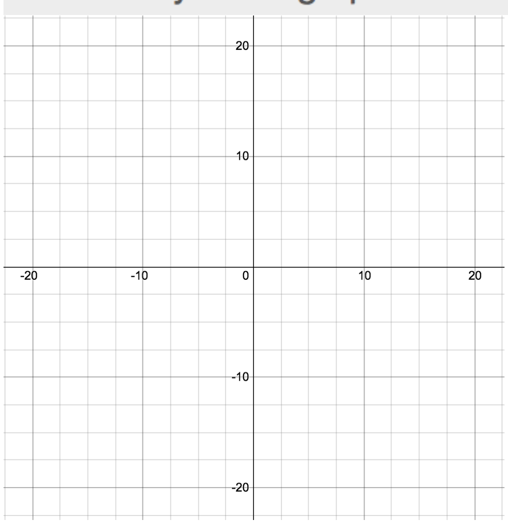



**c.**

The function $G(m)$ gives the number of gallons of gas required to drive $m$ miles. Interpret $G(m) + 10$ and $G(m + 10)$.

**d.**

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?



**Combining Vertical and Horizontal Shifts**

HOW TO

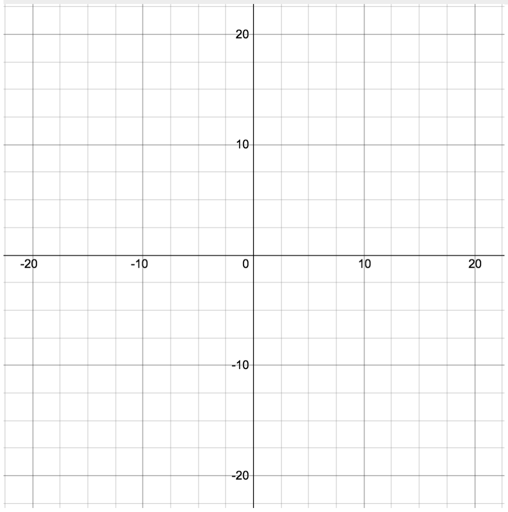
**Given a function and both a vertical and a horizontal shift, sketch the graph.**

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

**Example**

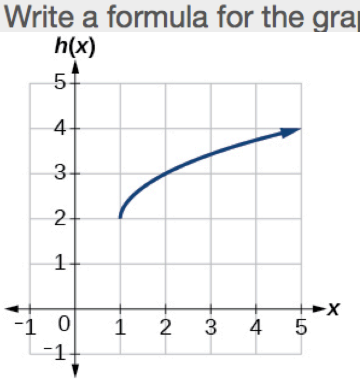
**a.**

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.




**b.**

Write a formula for the graph shown in Figure, which is a transformation of the toolkit square root function.

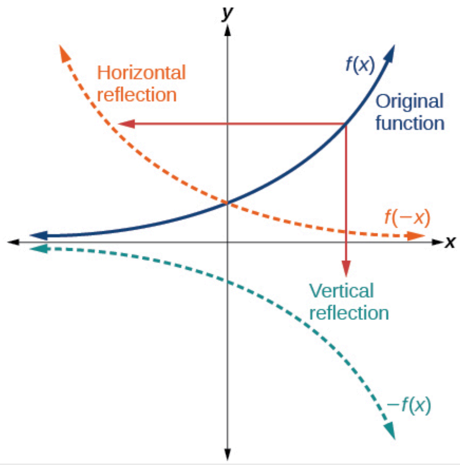



**c.**

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

**Graphing Functions Using Reflections About Axes**

Another transformation that can be applied to a function is a reflection over the *x*- or *y*-axis. A **vertical reflection** reflects a graph vertically across the ____-axis, while a **horizontal reflection** reflects a graph horizontally across the ____-axis. The reflections are shown in Figure.




## A GENERAL NOTE: REFLECTIONS

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the *y*-axis.

## HOW TO

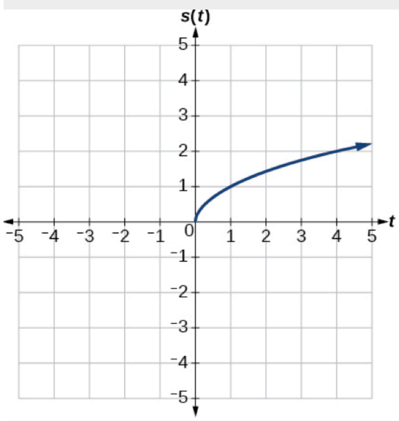
**Given a function, reflect the graph both vertically and horizontally.**

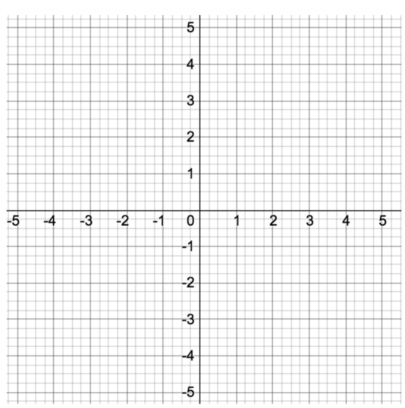
1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

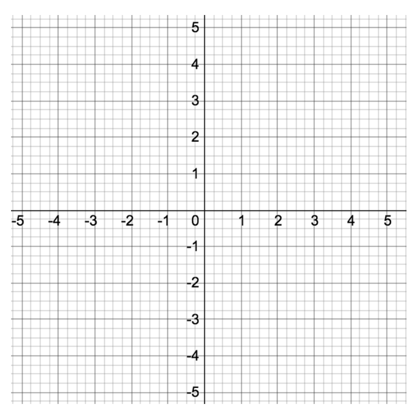
**Example**

**a.**

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.





**b.**

A function $f(x)$ is given as Table. Create a table for the functions below.

a. $g(x) = -f(x)$

b. $h(x) = f(-x)$

| $x$ | –2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 10 | 15 | 20 |

**Even and Odd Functions**

A GENERAL NOTE: EVEN AND ODD FUNCTIONS

A function is called an **even function** if for every input $x$

$$f(x) = f(-x)$$

The graph of an even function is symmetric about the $y$-axis.

A function is called an **odd function** if for every input $x$

$$f(x) = -f(-x)$$

The graph of an odd function is symmetric about the origin.

HOW TO

**Given the formula for a function, determine if the function is even, odd, or neither.**

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

**Example**

**a.** Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

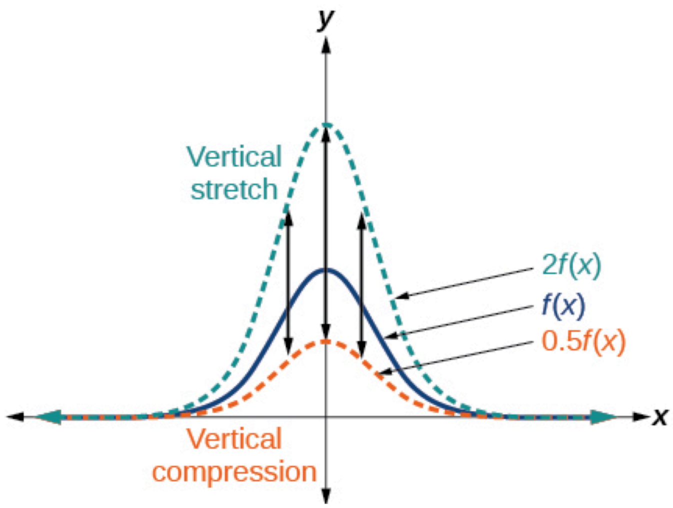
**b.** Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

**Graphing Functions Using Stretches and Compressions**

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity. We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

**Vertical Stretches and Compressoins**

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is ______________________ than 1, we get a ________________________ ____________________; if the constant is between ___ and ___ , we get a ________________________ ____________________. Figure shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.




## A GENERAL NOTE: VERTICAL STRETCHES AND COMPRESSIONS

Given a function $f(x)$, a new function $g(x) = af(x)$, where $a$ is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

## HOW TO

**Given a function, graph its vertical stretch.**

1. Identify the value of $a$.
2. Multiply all range values by $a$.
3. 

   If $a > 1$, the graph is stretched by a factor of $a$.

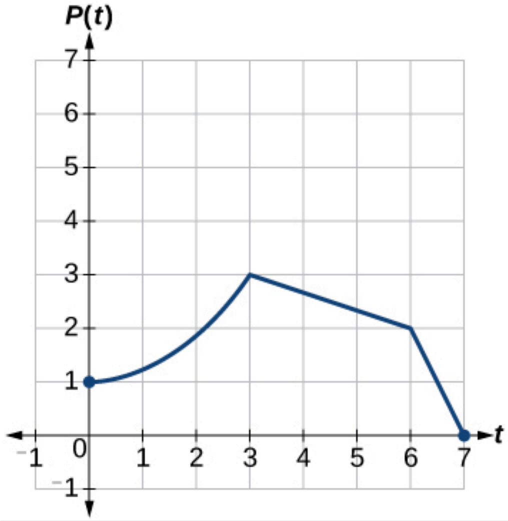
   If $0 < a < 1$, the graph is compressed by a factor of $a$.

   If $a < 0$, the graph is either stretched or compressed and also reflected about the $x$-axis.

**Example**

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure.

A scientist is comparing this population to another population, $Q$, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.




## HOW TO

**Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.**

1. Determine the value of $a$.
2. Multiply all of the output values by $a$.

**Example**

**a.**

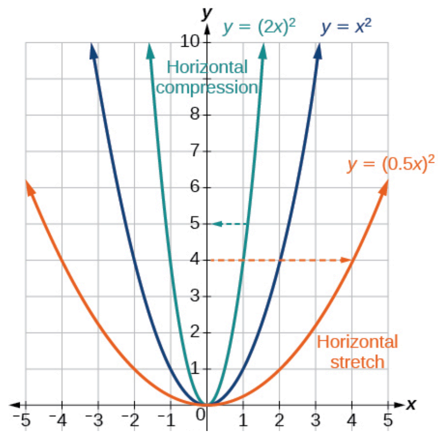
A function $f$ is given as Table. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 12 | 16 | 20 | 0 |

**b.** Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

**Horizontal Stretches and Compressions**

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between _____ and _____, we get a **horizontal _______________**; if the constant is greater than _____, we get a **horizontal _____________________________** of the function.




## A GENERAL NOTE: HORIZONTAL STRETCHES AND COMPRESSIONS

Given a function $f(x)$, a new function $g(x) = f(bx)$, where $b$ is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

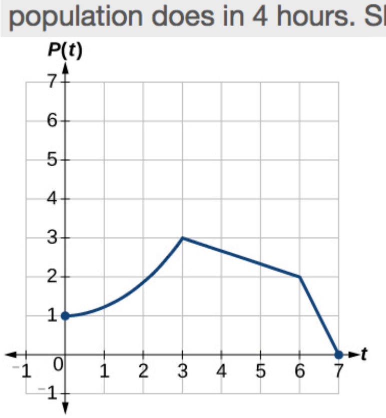
## HOW TO

**Given a description of a function, sketch a horizontal compression or stretch.**

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

**Example**

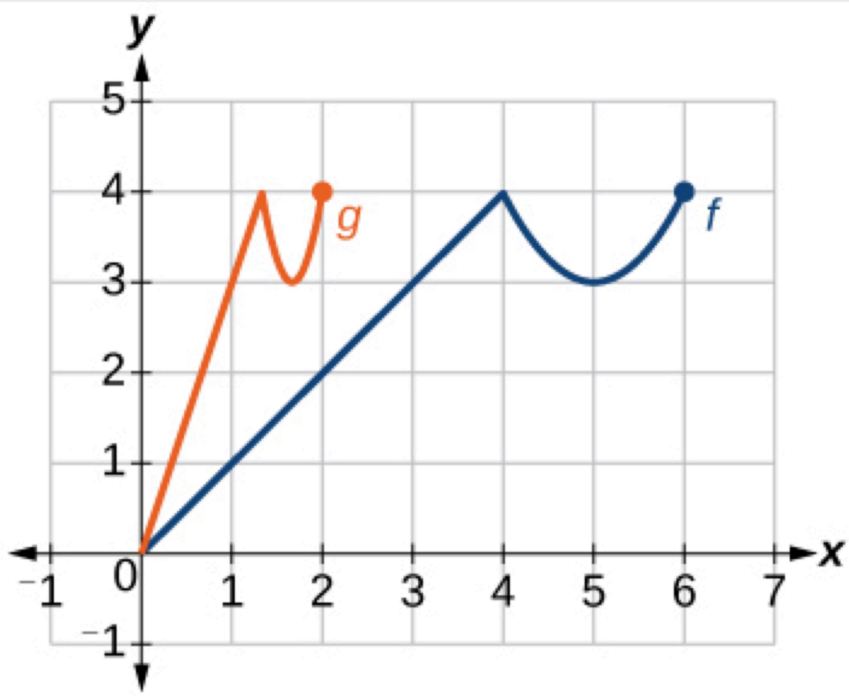
Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, $R$, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.



A function $f(x)$ is given as Table. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

Relate the function $g(x)$ to $f(x)$ in Figure.




Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

**Performing a Series of Transformations**

A GENERAL NOTE: COMBINING TRANSFORMATIONS

When combining vertical transformations written in the form $af(x) + k$, first vertically stretch by $a$ and then vertically shift by $k$.

When combining horizontal transformations written in the form $f(bx + h)$, first horizontally shift by $h$ and then horizontally stretch by $\frac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x + h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by $h$.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

## Order of function transformations

- Horizontal shifts
- Horizontal stretch/compression
- Reflection over y-axis
- Vertical stretch/compression
- Reflection over x-axis
- Vertical shifts
- Important note: Horizontal and vertical dilations are applied only to the portion of the function which is shifted horizontally. You NEVER multiply the dilation factors with vertical shifts.

**Example**

Given Table for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

| $x$ | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $f(x)$ | 10 | 14 | 15 | 17 |

Use the graph of $f(x)$ in Figure to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.