4.1 – Linear Functions

There are many situations that involve constant change over time. We will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations.

Representing Linear Functions

A ___________________________________ function, which is defined as a function with a constant rate of change. This is a polynomial of degree ____. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form.

Example

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time?

Representing a Linear Function in Word Form.

The train's ______________________________ from the station is a function of the ________________ during which the train moves at a _____________________________ speed plus its _______________________ distance from the station when it began moving at constant speed.

> The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the _____________-______________________________ form of a line, where ____ is the input value, ____ is the rate of change, and ____ is the initial value of the dependent variable.

Equation form	$y = mx + b$
Function notation	$f(x) = mx + b$

In the example of the train, we might use the notation $D(t)$ where the total distance D is a function of the time t.

Representing a Linear Function in Tabular Form

t	**0**	**1**	**2**	**3**
D(t)				

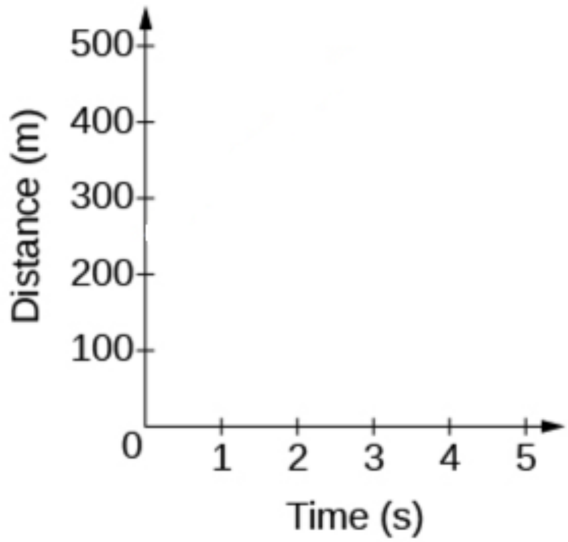
Q&A

Can the input in the previous example be any real number?

No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

When we plot a linear function, the graph is always a ________________. The rate of change, which is constant, determines the slant, or __________________ of the line. The point at which the input value is zero is the vertical intercept, or ___-intercept, of the line. We can see from the graph that the *y*-intercept in the train example we just saw is (0,250) and represents the distance of the train from the station when it began moving at a constant speed.



A GENERAL NOTE: LINEAR FUNCTION

A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The *y*-intercept is at $(0, b)$.

Example

The pressure, P,in pounds per square inch (PSI) on the diver in Figure depends upon her depth below the water surface, d,in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

A GENERAL NOTE: THE SLOPE OF A LINE

The slope of a line, m, represents the change in y over the change in x. Given two points, (x_1, y_1) and (x_2, y_2) ,the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a __________________________________ slope. A line with a positive slope slants ____________________ from left to right as in Figure**(a)**. For a decreasing function, the slope is _________________________. The output values decrease as the input values increase. A line with a negative slope slants ________________________ from left to right as in Figure**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure**(c)**.

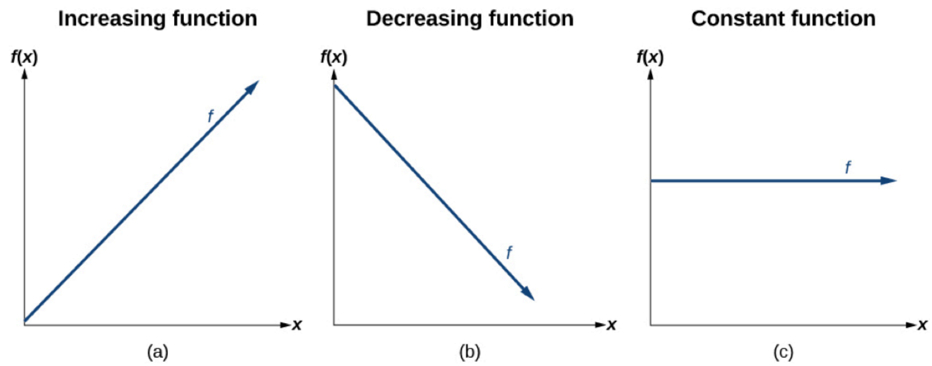


Figure 5.

A GENERAL NOTE: INCREASING AND DECREASING FUNCTIONS

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

$f(x) = mx + b$ is an increasing function if $m > 0$.
$f(x) = mx + b$ is a decreasing function if $m < 0$.
$f(x) = mx + b$ is a constant function if $m = 0$.

Example

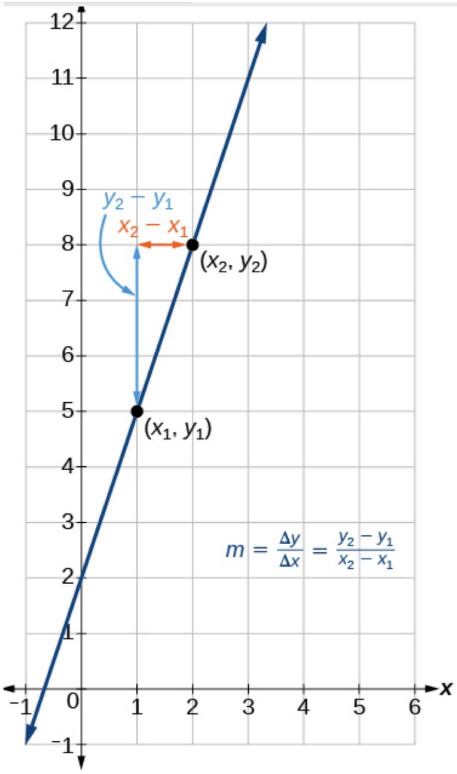
Some recent studies suggest that a teenager sends an average of 60 tests per day. For each of the following scenarios find the linear function that describes the relationship between the input and the output value. Then determine whether the graph of the function is increasing, decreasing, or constant.

a. **The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and the output is the total number of texts.**
b. **A teen has a limit of 500 texts per month on his/her data plan. The input is the number of days, and output is the total number of texts remaining that month.**
c. **A teem has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and the output is the total cost of texting each month.**

Interpreting Slope as a Rate of Change

In the examples we have seen so far, the slope was provided to us. However, we often need to calculate the slope given input and output values. Recall that given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Q&A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

HOW TO

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Writing and Interpreting an Equation for a Linear Function

A GENERAL NOTE: THE POINT-SLOPE FORMULA

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1)$$

A GENERAL NOTE: LINEAR FUNCTION

A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The *y*-intercept is at $(0, b)$.

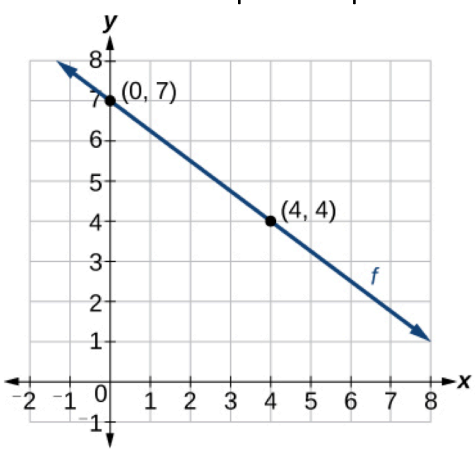
HOW TO

Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the *y*-axis to identify the *y*-intercept by visual inspection.
4. Substitute the slope and *y*-intercept into the slope-intercept form of a line equation.

Example

a. Write the point-slope and slope-intercept form of the equation for the graphed line.



b. Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

c. If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, write an equation for the function in slope-intercept form.

Modeling Real-World Problems with Linear Functions

HOW TO

Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Examples

a. Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies and earned $920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

b. Table relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

c. A new plant food was introduced to a young tree to test its effect on the height of the tree. Table shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
H(x)	12.5	13.5	14.5	16.5	18.5

Graphing Linear Functions

Graphing a Function By Plotting Points

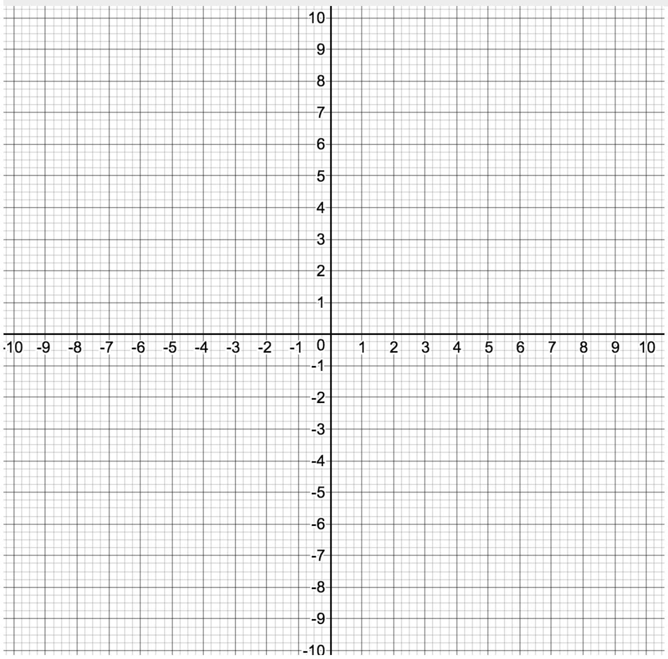
HOW TO

Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Example

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.



Graphing a Linear Function Using y-intercept and Slope

A GENERAL NOTE: GRAPHICAL INTERPRETATION OF A LINEAR FUNCTION

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

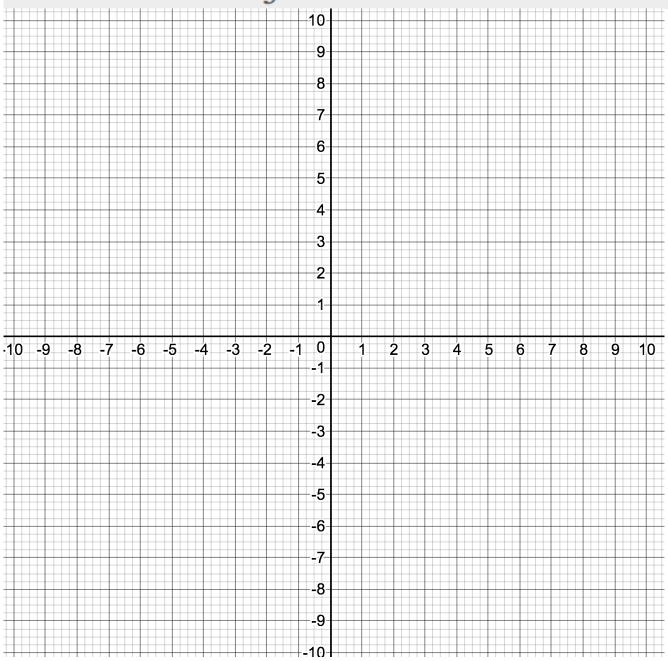
HOW TO

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

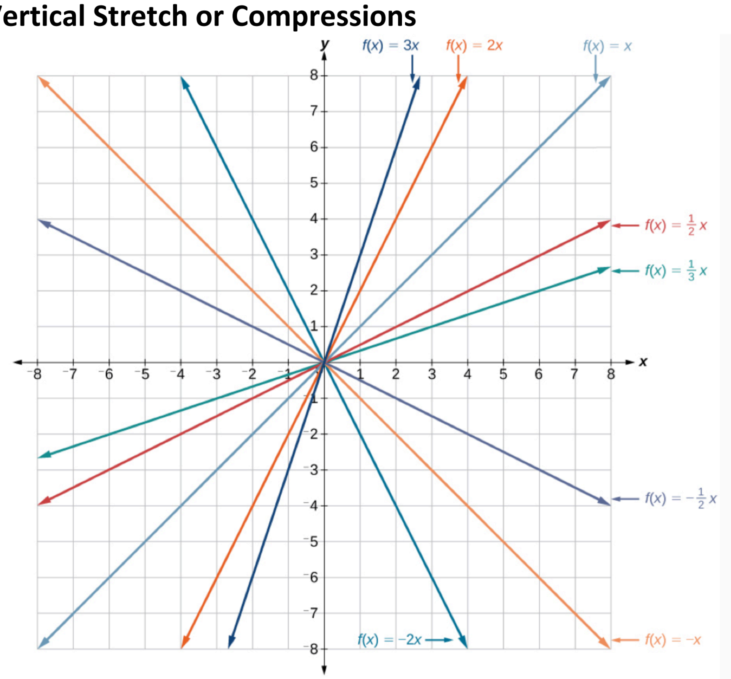
Example

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

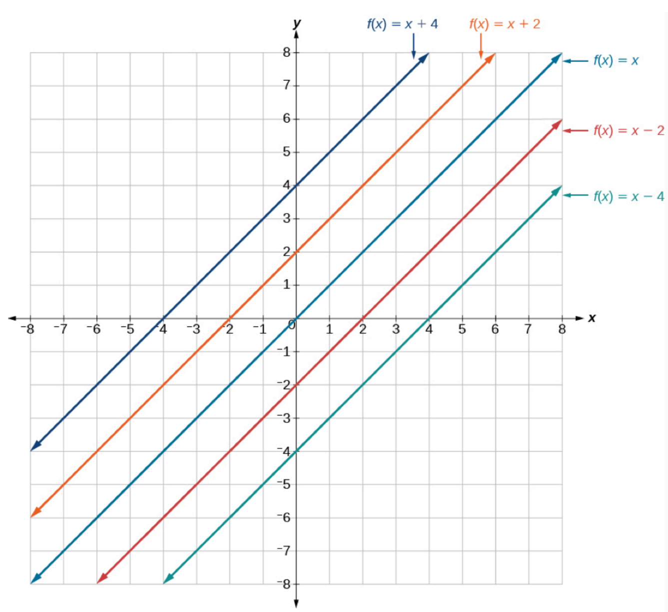


Graphing a Function Using Transformations

Vertical Stretch or Compressions

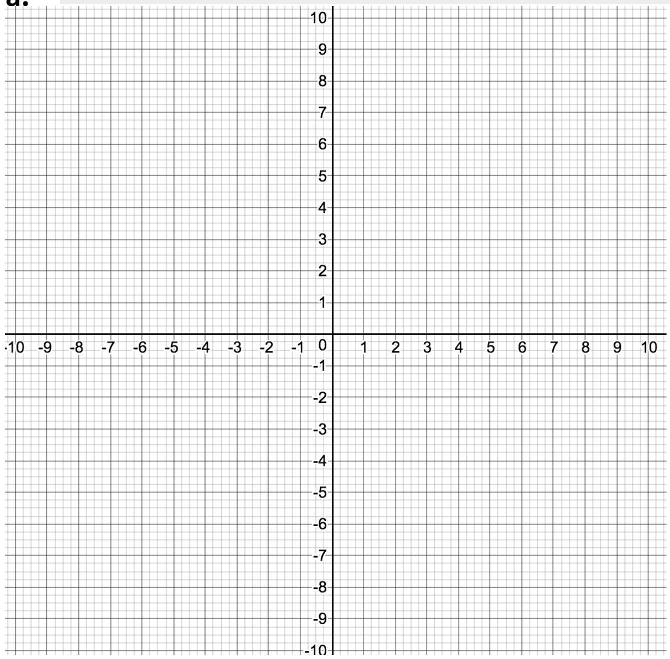


Vertical Shift

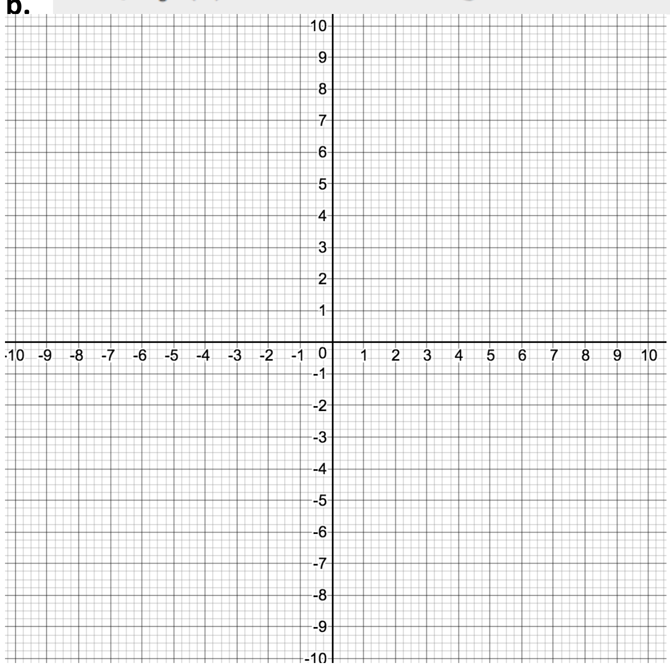


Example

a. Graph $f(x) = \frac{1}{2}x - 3$ using transformations.



b. Graph $f(x) = 4 + 2x$ using transformations.



Writing the Equation for a Function from the Graph of a Line

HOW TO

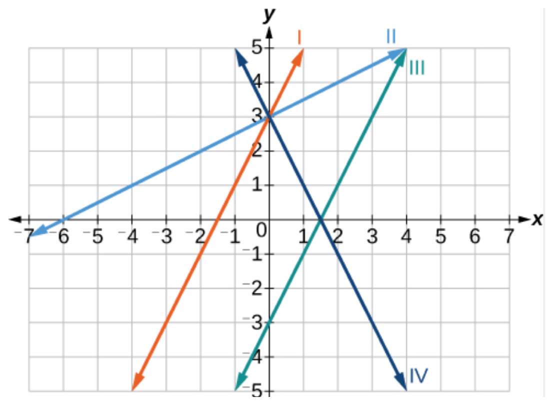
Given a graph of linear function, find the equation to describe the function.

1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

Example

Match each equation of the linear functions with one of the lines in Figure.

$$\begin{array}{llcl} a. & f(x) & = & 2x+3 \\ b. & g(x) & = & 2x-3 \\ c. & h(x) & = & -2x+3 \\ d. & j(x) & = & \frac{1}{2}x+3 \end{array}$$



Finding the x-intercept of a Line

A GENERAL NOTE: X-INTERCEPT

The *x*-intercept of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example

Find the *x*-intercept of $f(x) = \frac{1}{4}x - 4$.

Find the *x*-intercept of $f(x) = \frac{1}{2}x - 3$.

Describing Horizontal and Vertical Lines

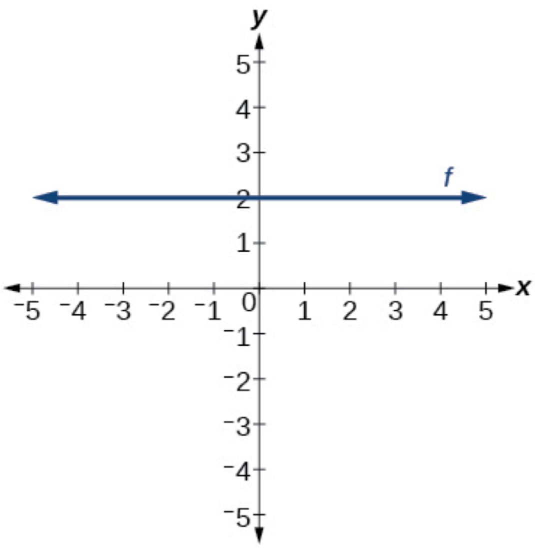
A GENERAL NOTE: HORIZONTAL AND VERTICAL LINES

Lines can be horizontal or vertical.

A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

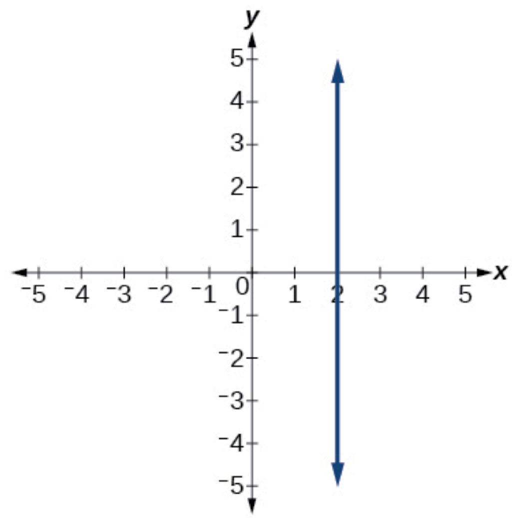
A **vertical line** is a line defined by an equation in the form $x = a$.

There are two special cases of lines on a graph—horizontal and vertical lines. A ___________________________________line indicates a constant output, or *y*-value. A ______________________ line indicates a constant input, or *x*-value.



x	-4	-2	0	2	4
y	2	2	2	2	2

Figure 17. A horizontal line representing the function $f(x) = 2$



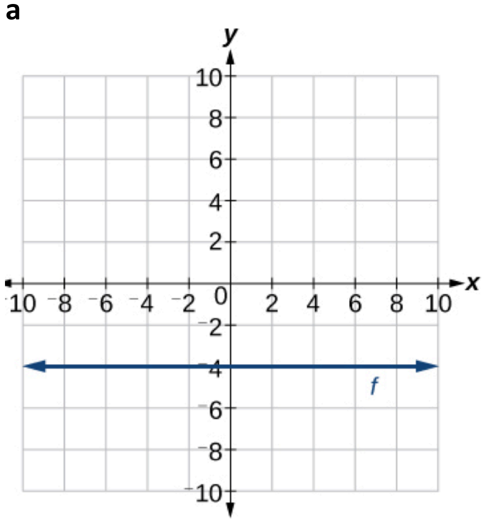
x	2	2	2	2	2
y	-4	-2	0	2	4

Figure 19. The vertical line, $x = 2$, which does not represent a function

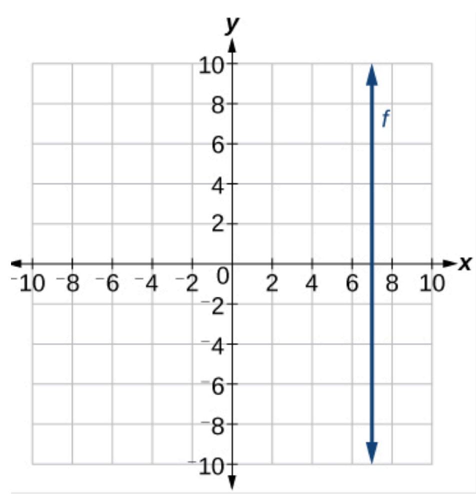
Example

Write the equation of each of the following lines.

a

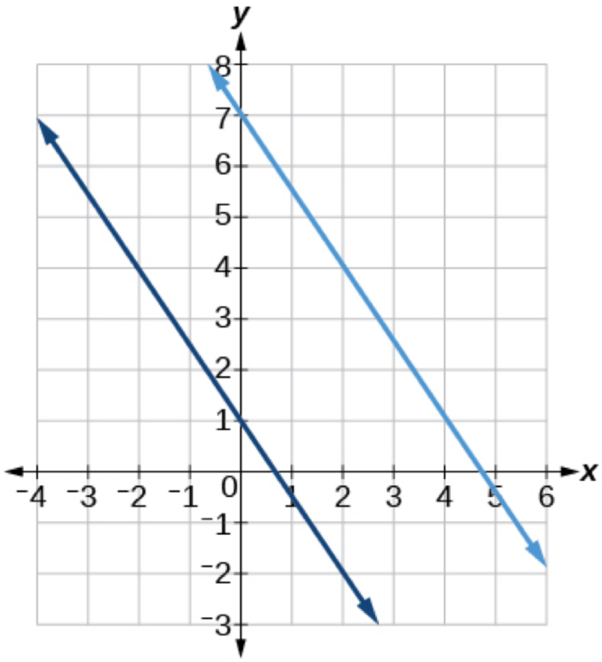


b.



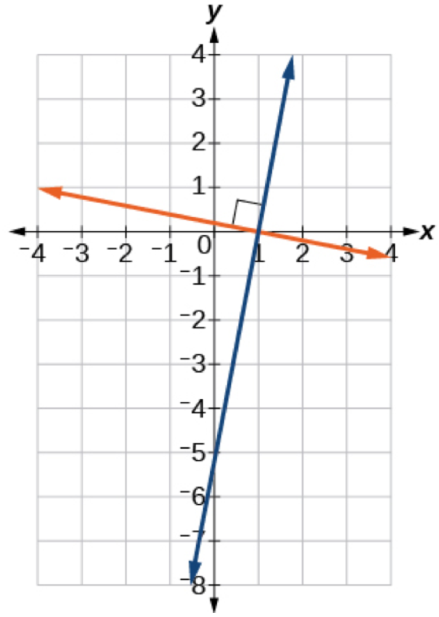
Determining Whether Lines are Parallel or Perpendicular

If two lines are ______________________ lines, they will never intersect. They have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the *y*-intercept. If we shifted one line vertically toward the other, they would become coincident.



______________________________ lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield -1.

$$m_1 m_2 = -1$$



A GENERAL NOTE: PARALLEL AND PERPENDICULAR LINES

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$$f(x) = m_1 x + b_1 \text{ and } g(x) = m_2 x + b_2 \text{ are parallel if and only if } m_1 = m_2$$

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect to form a right angle.

$$f(x) = m_1 x + b_1 \text{ and } g(x) = m_2 x + b_2 \text{ are perpendicular if and only if}$$

$$m_1 m_2 = -1, \text{ so } m_2 = -\frac{1}{m_1}$$

Example

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$f(x) = 2x + 3 \qquad h(x) = -2x + 2$$
$$g(x) = \tfrac{1}{2}x - 4 \qquad j(x) = 2x - 6$$

Writing the Equation of Parallel and Perpendicular Lines

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x) = mx + b$.
4. Solve for b.
5. Write the equation of the line.

Example

Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0, 0)$ that is

a. parallel to $h(x)$
b. perpendicular to $h(x)$

A line passes through the points, $(-2, -15)$ and $(2, -3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.