### Hyperbolas KEY

### Objectives

* Graph a hyperbola with center at $\left(0,0\right)$
* Identify conic sections by their equations

### Graph a Hyperbola with Center at (0, 0)

The last conic section we will look at is called a hyperbola. We will see that the equation of a hyperbola looks similar to the equation of an **ellipse**, except it is a **difference** rather than a sum.

Hyperbola

A **hyperbola** is all points in a plane where the **difference** of their distances from two fixed points is **constant**. Each of the fixed points is called a **focus** of the hyperbola.



Definitions:

* The line through the foci is called the **transverse axis**.
* The two points where the transverse axis intersects the hyperbola are each a **vertex** of the hyperbola.
* The midpoint of the segment joining the foci is called the **center** of the hyperbola.
* The line perpendicular to the transverse axis that passes through the center is called the **conjugate axis**.
* Each piece of the graph is called a **branch** of the hyperbola.



Standard Form of the Equation a Hyperbola with Center $\left(0,0\right)$

The standard form of the equation of a hyperbola with center $\left(0,0\right),$ is given two ways:

$$\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1   or   \frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$$



* **A** is the distance from the center to **vertices** along the **transverse** axis.
* **B** is the distance from the center to the points on the **conjugate** axis, used to determine how **wide** the hyperbola opens.
* **C** is the distance from the center to the **foci**.
* Pythagorean relationship: $c^{2}=a^{2}+b^{2}$**.**
* The **transverse axis** contains the vertices and foci and is determined by which term is **positive**.

**Asymptotes**

The **asymptotes** are intersecting straight lines that the branches of the graph **approach** but never intersect as the *x*, *y* values get larger and larger. The asymptotes are determined by the values of ***a* and *b***.



To find the asymptotes, we sketch a **rectangle** whose sides are based off the values of *a* and *b*. The lines containing the **diagonals** of this rectangle are the asymptotes of the hyperbola. The rectangle and asymptotes are not part of the hyperbola, but they help us sketch the graph.

1. Graph the hyperbola.

 $\frac{x^{2}}{25}-\frac{y^{2}}{4}=1$

           

1. Graph the hyperbola.

 $4y^{2}-16x^{2}=64$

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| **To write the equation in standard form, divide each term by 64 to make the equation equal to 1.** | $$\frac{4y^{2}}{64}-\frac{16x^{2}}{64}=\frac{64}{64}$$ |
| **Simplify.** | $$ \frac{y^{2}}{16}-\frac{x^{2}}{4}=1$$ |
| **Since the *y*2-term is positive, the transverse axis is vertical. Since** $a^{2}=16$ **then** $a=\pm 4.$ |  |
| **The vertices are on the *y*-axis,** $\left(0,-a\right),$$\left(0,a\right). $**Since** $b^{2}=4$ **then** $b=\pm 2.$ | **The vertices are** $\left(0,-4\right),$$\left(0,4\right).$ |
| **The foci are on the y-axis, (0, -c), (0, c). Since** $c^{2}=a^{2}+b^{2}$**, we have that** $c^{2}=4+16$**=20. Therefore,** $c=\pm 2\sqrt{5}$**.**  | **The foci are** $\left(0,-2\sqrt{5}\right), (0, 2\sqrt{5})$**.** |
| **Sketch the rectangle intersecting the *x*-axis at** $\left(-2,0\right),$$\left(2,0\right)$ **and the *y*-axis at the vertices.Sketch the asymptotes through the diagonals of the rectangle. Draw the two branches of the hyperbola.** | **.** |

1. Find an equation of a hyperbola centered at the origin with a vertex at (0,3) and a focus at (0,-6). Then sketch the graph.

**The transverse axis is the y-axis, so the equation will be of the form** $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$**. From what is given, we have that** $a=3$ **and** $c=6$**. To find** $b$**, use the Pythagorean relationship. We know that** $c^{2}=a^{2}+b^{2}$**, meaning that** $36-9=b^{2}$ **and so** $b^{2}=27 or b=\pm 3\sqrt{3}$**. Therefore, the equation is given by** $\frac{y^{2}}{9}-\frac{x^{2}}{27}=1$**.**



### Identify Conic Sections by their Equations

To identify a conic from its equation, put the **variable** terms on one side of the equation and the **constants** on the other. It may be helpful to put the equations in **standard form**.

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|  |  | Example |
| **Parabola** | Either $x^{2}$ OR $y^{2}.$ Only **one** variable is squared. | $$x=3y^{2}-2y+1$$ |
| **Circle** | $x^{2}-$ and $y^{2}-$ terms have the **same** **coefficients** | $$x^{2}+y^{2}=49$$ |
| **Ellipse** | $x^{2}-$ and $y^{2}-$ terms have the **same** **sign**, different coefficients | $$4x^{2}+25y^{2}=100$$ |
| **Hyperbola** | $x^{2}-$ and $y^{2}-$ terms have **different signs**, different coefficients | $$25y^{2}-4x^{2}=100$$ |

Identify the graph of each equation as a circle, parabola, ellipse, or hyperbola. Then put the equation in standard form

1. $9x^{2}=72-36y^{2}$

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| --- | --- |
|  | $$9x^{2}+36y^{2}=72$$ |
| **The** $x^{2}$**- and** $y^{2}$**-terms have the same sign and different coefficients.** | **Ellipse** |
| **To put in standard form, divide everything by 72.** | $$\frac{9x^{2}}{72}+\frac{36y^{2}}{72}=\frac{72}{72}$$ |
| **Simplify.** | $$\frac{x^{2}}{8}+\frac{y^{2}}{2}=1$$ |

1. $x^{2}+y^{2}-6x-8y=0$

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| --- | --- |
|  | $$x^{2}+y^{2}-6x-8y=0$$ |
| **The** $x^{2}$**- and** $y^{2}$**-terms have the same coefficients.** | **Circle** |
| **To put in standard form, complete the square.**$$x^{2}-6x+9+y^{2}-8y+16=0+9+16$$ | $$\left(x-3\right)^{2}+\left(y-4\right)^{2}=25$$ |

1. $y+4x=-2x^{2}-5$

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|  | $$y=-2x^{2}-4x-5$$ |
| Only one variable, $x$, is squared. | Parabola |
| To put in standard form, complete the square.$$y=-2\left(x^{2}+2x+1\right)-5+2$$$$y=-2\left(x+1\right)^{2}-3$$ | $$-\frac{1}{2}\left(y+3\right)=\left(x+1\right)^{2}$$ |

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