8.5 Systems of Non-linear Equations in Two Variables

Often, we are interested in finding the places where different graphs intersect each other because this gives us useful information about the corresponding relations. For example, in calculus, we need endpoints for integration to find the areas of regions that the graphs contain. This is helpful to find the center of mass in physics and engineering applications.

To solve a system of non-linear equations in two variables, the general method of substitution is the approach we most often need. By solving each equation for the same variable, typically 'y', and then setting the expressions equal to each other, we can arrive at a single equation in a single variable. Of course, we can also solve one of the equations for one of the variables and substitute that expression into the other equation to arrive at a single variable equation.

For example, if we have the two equations $y = x^2 + 2x$ (a parabola) and $y = 3x + 2$ (a line), we can find the intersection points of these two graphs by setting them equal to each other and solving to get the x-values of those points:

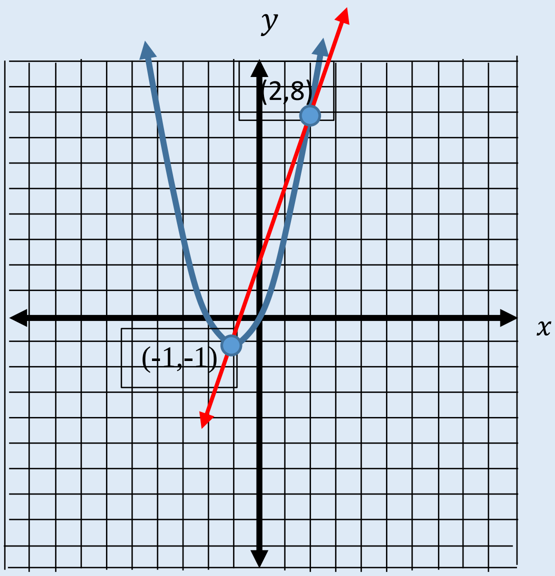
$$x^2 + 2x = 3x + 2$$

We recognize this as a polynomial equation, specifically a quadratic, so we bring everything to one side to get 0 on one side and see if we can factor (or use the quadratic formula). It is very important to be able to identify the type of equation you have in order to know which technique to use to solve it. Here, we obtain:

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x - 2 = 0; \;\; x + 1 = 0$$
$$x = 2; \;\; x = -1$$

We have found the x-values of the points and we could plug these into either equation to get the corresponding y-values. It is also very helpful to graph both relations to see a picture of the intersections and interpret the meaning and validity of solutions.

For $x = 2$, we obtain $y = 3(2) + 2 = 8$, which gives us the point $(2, 8)$. For $x = -1$, we have $y = 3(-1) + 2 = -1$, which gives us the point $(-1, -1)$.

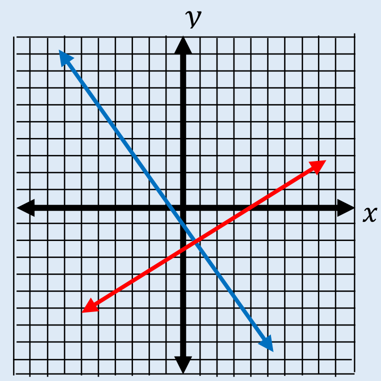


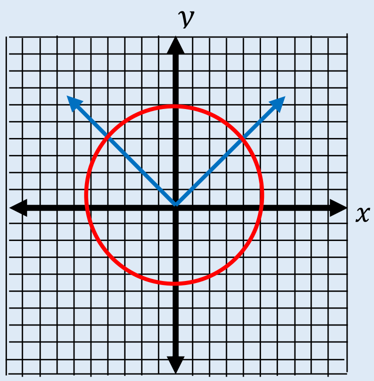
To graph the line, you could use the slope and y-intercept, since the equation of the line is already in slope-intercept form. To graph the parabola, you could complete the square to find the vertex and then plug in one other point and use symmetry. If you complete the square on the parabola, you should get

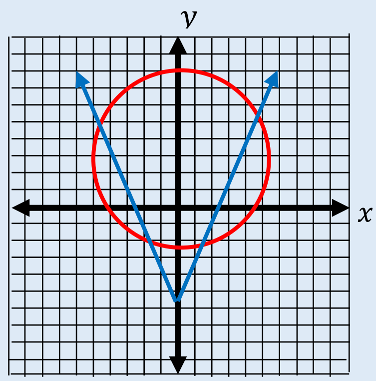
$y = (x + 1)^2 - 1$ (Review Sections 2.8 and 6.1 if needed).

Now that we can see the meaning of solving a non-linear system of equations is just finding the intersection points of the corresponding graphs, we can imagine many examples that include a combination of the graphs we have learned.

Some Possibilities:



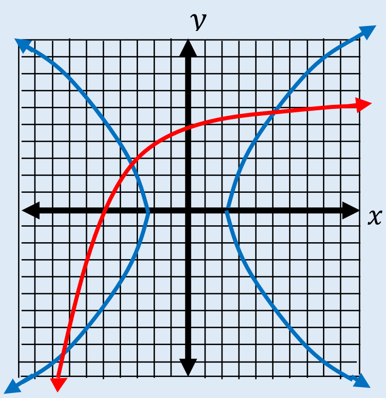


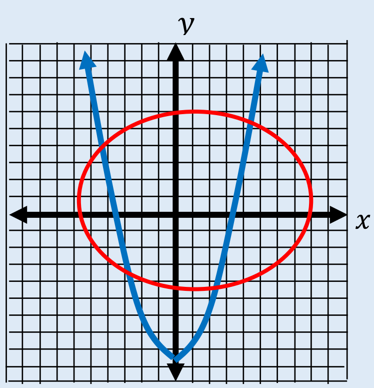


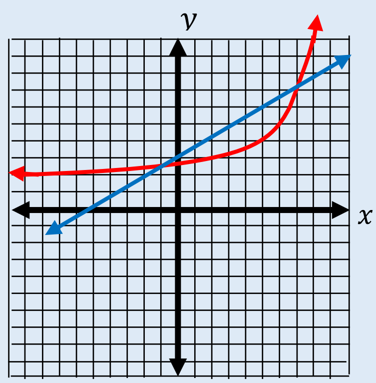
Two lines (we did this in Section 8.1 with linear systems) with one intersection.

An absolute value function and a circle with two intersection points.

An absolute value function and a circle with four intersection points.







A logarithmic function and a hyperbola with three intersection points.

A parabola and an ellipse with four intersection points.

A line and an exponential function with two intersection points.

Some of the systems above would require more advanced equation solving methods than we have done in this class, but these examples do give you the big picture. We will restrict our future examples to ones with resulting equations that can be solved by the methods of this course. You will learn about numerical methods to solve more complicated equations in future courses, such as calculus. But there are many that we already have the skills to solve!

Examples

Find the intersection points for each pair of relations. Draw the graphs together on the same set of coordinate axes to see that your solution makes sense.

1. $y = x^2$; $y = 3x + 4$

First, solve the system: $y = x^2$; $y = 3x + 4$

$$x^2 = 3x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x = 4; x = -1$$

Now, we can find the corresponding y-values. We can use either of the original equations since the points are common to both. The parabola is easy to use here.

For $x = 4$,

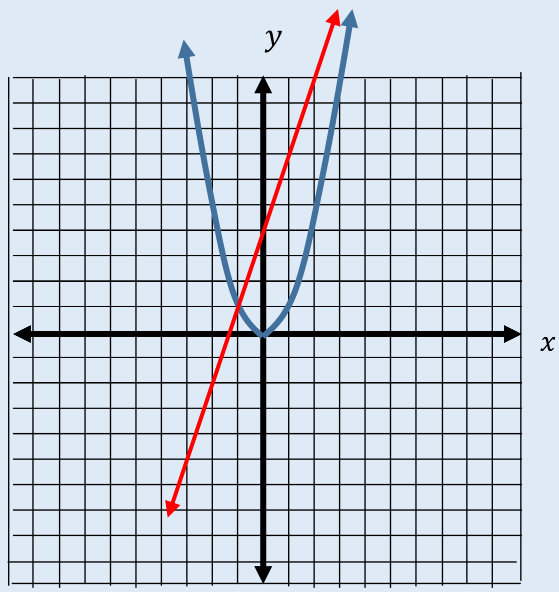
$$y = 4^2 = 16$$

For $x = -1$,

$$y = (-1)^2 = 1$$

So, the solutions are $(\mathbf{4},\mathbf{16})$ and $(\mathbf{-1},\mathbf{1})$. Looking at the graph, we will see that these are reasonable solutions. (Sometimes checking the graph is how you can find algebraic errors).

It is important to recognize the type of graph that goes with each equation to know the method to graph each. Here, we have a parabola and a line. We use the slope $m = 3$ and the y-intercept $(0,4)$ to graph the line. We can graph the parabola by plotting the vertex $(0,0)$ and then one other point and use symmetry for a final point. This one is particularly easy as it is the basic parabola graph.



After graphing both, we see that there are two intersection points. We can also see about where they should be to see if our answers make sense.

2. $x^2 + y^2 = 16;\ \ x + 3y = 4$

Using a simple substitution is easiest here. Solving the second equation for x, we get $x = 4 - 3y$. Now, if we substitute this into the first equation, we get:

$$(4 - 3y)^2 + y^2 = 16$$

$$16 - 24y + 10y^2 = 16$$
$$10y^2 - 24y = 0$$
$$2y(5y - 12) = 0$$
$$y = 0; y = \frac{12}{5}$$

Now, we can find the corresponding x-values. Remember, we can use either of the original equations to find these. The line is easy to use here and we already have it solved for x $(x = 4 - 3y)$, so we can use that form for ease.

For $y = 0$,

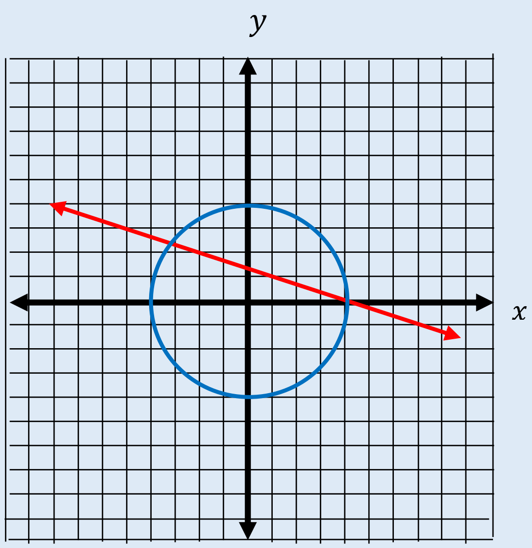
$$x = 4 - 3(0) = 4$$

For $y = \frac{12}{5}$,

$$x = 4 - 3\left(\frac{12}{5}\right) = \frac{20-36}{5} = -\frac{16}{5}$$

So, the solutions are $(4, 0)$ and $\left(-\frac{16}{5}, \frac{12}{5}\right)$.

Here, we have a circle centered at $(0, 0)$ of radius 4 and a line.



The solutions make sense looking at the graph. If you rewrite the second solution as a pair of mixed numbers instead, it is easy to see: $(-3\frac{1}{5}, 2\frac{2}{5})$.

3. $y = |x|;\ \ x + 2y = 5$

Using a simple substitution is easiest here. If we substitute $|x|$ from the first equation into the second equation for y, we get:

$$x + 2|x| = 5$$
$$2|x| = 5 - x$$
$$|x| = \frac{5 - x}{2}$$

Notice the resulting equation has absolute value, so we must isolate the absolute value and split into two equations.

$$x = \frac{5-x}{2} \qquad\qquad x = -\left(\frac{5-x}{2}\right)$$
$$2x = 5 - x \qquad\qquad -2x = 5 - x$$
$$3x = 5 \qquad\qquad -x = 5$$
$$x = \frac{5}{3} \qquad\qquad x = -5$$

Now, we can find the corresponding y-values. The absolute value function is easy to use here for this purpose.

For $x = 5/3$,

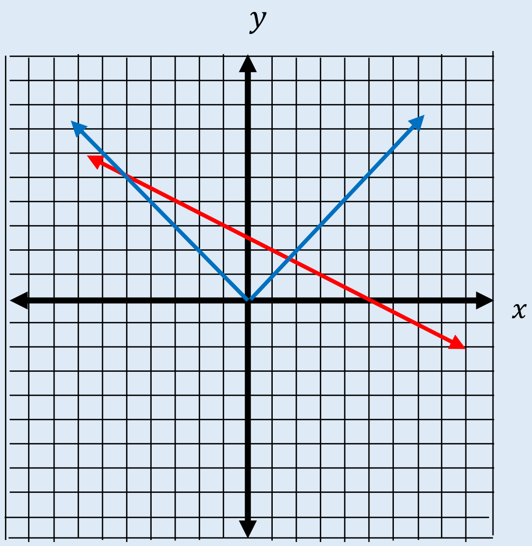
$$y = \left|\frac{5}{3}\right| = \frac{5}{3}$$

For $x = -5$,

$$y = |-5| = 5$$

So, the solutions are $\left(\frac{5}{3}, \frac{5}{3}\right)$ and $(-5, 5)$.

Graphing the absolute value function and the line, we obtain the following picture:



The solutions make sense looking at the graph. If you rewrite the first solution as a pair of mixed numbers instead, it is easy to see: $(1\frac{2}{3}, 1\frac{2}{3})$.

4. $y = \sqrt{x+2}$; $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Since the first equation is already solved for y, it is convenient to substitute this into the second equation, we get:

$$\frac{x^2}{4} - \frac{\left(\sqrt{x+2}\right)^2}{9} = 1$$

$$\frac{x^2}{4} - \frac{x+2}{9} = 1$$

$$9x^2 - 4(x+2) = 36$$

$$9x^2 - 4x - 8 = 36$$

$$9x^2 - 4x - 44 = 0$$

$$(x+2)(9x-22) = 0$$

$$x = -2; x = \frac{22}{9}$$

To find the corresponding y-values, the radical equation appears the simpler choice.

For $x = -2$,

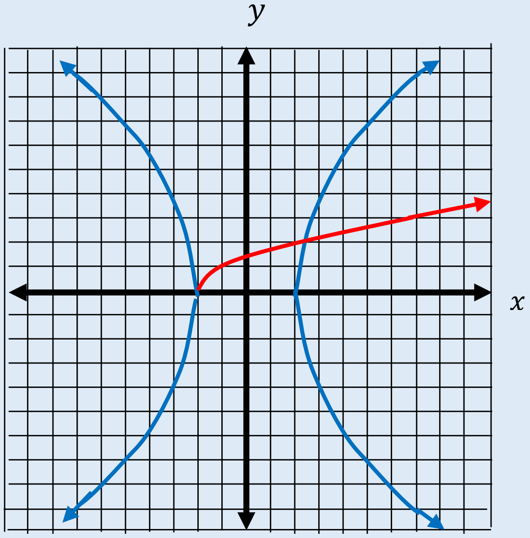
$$y = \sqrt{-2 + 2} = \sqrt{0} = 0$$

For $x = \frac{22}{9}$,

$$y = \sqrt{\frac{22}{9} + 2} = \sqrt{\frac{22+18}{9}} = \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}$$

So, the solutions are $(-2, 0)$ and $\left(\frac{22}{9}, \frac{2\sqrt{10}}{3}\right)$.

Here, we have a hyperbola centered at $(0, 0)$ with $a = 2$ and $b = 3$ and a basic square root function shifted to the left by 2.



The solutions make sense looking at the graph. If you rewrite the second solution as a pair of decimals instead, it is easier to see: (2.44 , 2.11).

5. $y = \frac{1}{x-1}$; $y = 2x + 4$

Since the first equation is already solved for y, it is convenient to substitute this into the second equation, we get:

$$\frac{1}{x-1} = 2x + 4$$

$$1 = (2x + 4)(x - 1)$$

$$1 = 2x^2 + 2x - 4$$

$$0 = 2x^2 + 2x - 5$$

We need to use the quadratic formula here since we cannot factor.

$$a = 2; b = 2; c = -5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{-2 \pm \sqrt{4 + 40}}{4}$$

$$x = \frac{-2 \pm \sqrt{44}}{4} = \frac{-2 \pm 2\sqrt{11}}{4} = \frac{-1 \pm \sqrt{11}}{2}$$

To find the corresponding y-values, the linear equation appears the simpler choice.

For $x = \frac{-1+\sqrt{11}}{2}$,

$$y = 2\left(\frac{-1+\sqrt{11}}{2}\right) + 4 = -1 + \sqrt{11} + 4 = 3 + \sqrt{11}$$

For $x = \frac{-1-\sqrt{11}}{2}$,

$$y = 2\left(\frac{-1-\sqrt{11}}{2}\right) + 4 = -1 - \sqrt{11} + 4 = 3 - \sqrt{11}$$

So, the solutions are $\left(\frac{-1+\sqrt{11}}{2}, 3 + \sqrt{11}\right)$ and $\left(\frac{-1-\sqrt{11}}{2}, 3 - \sqrt{11}\right)$.

As decimals, these solutions are easier to interpret: $(1.16, 6.32)$ and $(-2.16, -0.32)$.

Here, we have a line with y-intercept $(0,4)$ and slope $m = 2$ and a basic rational function shifted to the right by 1.