8.4 Solving Linear Systems Using Determinants and Cramer's Rule

Cramer's Rule is yet another method we can use to solve systems of linear equations. It is a fairly fast method, too, if you are quick at computation! It does involve a new concept, called the determinant, however. Let's learn about determinants and then we can talk about Cramer's Rule.

Determinants only exist for $n \times n$ matrices.

same number of
rows and columns

Notation: The absolute value symbol is often used to denote the determinant of a matrix, although determinants can be negative:

$$|A| = \det A$$

For 2×2 matrices, the determinant is defined to be:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example:

$$\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = (2)(1) - (3)(2) = 2 - 6 = -4$$

What about a 3×3 matrix? We can expand the determinant (by *cofactors*) into 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Here $a_1, b_1, and\ \ c_1$ are the cofactors we used. If you cross out the row and column of the cofactor, you will be left with the matrix that you want the determinant for (the one next to the cofactor). You can use any row or column to obtain your cofactors, but you must alternate signs between them.

Examples

1.

$$\begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = -1 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\begin{aligned} &= -1(1-(-3)) - 2(2-(-3)) + 1(2-1) \\ &= -1(1+3) - 2(2+3) + 1(2-1) \\ &= -1(4) - 2(5) + 1(1) \\ &= -4 - 10 + 1 \\ &= -13 \end{aligned}$$

2.

$$\begin{vmatrix} 3 & 5 & 1 \\ 6 & -2 & 2 \\ 8 & -1 & 4 \end{vmatrix} = 3 \begin{vmatrix} -2 & 2 \\ -1 & 4 \end{vmatrix} - 5 \begin{vmatrix} 6 & 2 \\ 8 & 4 \end{vmatrix} + 1 \begin{vmatrix} 6 & -2 \\ 8 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 3(-8-(-2)) - 5(24-16) + 1(-6-(-16)) \\ &= 3(-8+2) - 5(24-16) + 1(-6+16) \\ &= 3(-6) - 5(8) + 1(10) \\ &= -18 - 40 + 10 \\ &= -48 \end{aligned}$$

We will introduce Cramer's Rule.

Cramer’s Rule

To solve a system of two equations with two unknowns, we will use several determinants to get values for x and y. We will use an x-determinant which will be labeled D_x, a y-determinant which will be labeled D_y, and a denominator determinant which will be labeled D. Then we will use those values in the formulas below to calculate the values of the variables x and y.

$$x = \frac{D_x}{D}; \quad y = \frac{D_y}{D}$$

For a 2×2 system of equations, we begin by rewriting it as an augmented matrix.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \Rightarrow \left| \begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right|$$

To calculate the denominator determinant, D, we will use the coefficients of x and y to create a 2×2 matrix.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

To calculate the x-determinant, we replace the column in the denominator determinant that corresponds to the coefficients of x (the first column) with the column that corresponds to the constants (the third column). The second column in the denominator determinant will remain unchanged. So, we will replace the column containing a and c with e and f.

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} = ed - bf$$

We will calculate the y-determinant similarly. Replace the column in the denominator determinant that corresponds to the coefficients of y (the second column) with the column that corresponds to the constants (the third column). The first column in the denominator determinant will remain unchanged. So, we will replace the column containing b and d with e and f.

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

We can see why this works using elimination on the 2×2 matrix.

$$\begin{array}{rl} d(ax + by = e) \Rightarrow & adx + bdy = ed \\ -b(cx + dy = f) \Rightarrow & \underline{-bcx - bdy = -bf} \\ & (ad - bc)x = ed - bf \\ & x = \dfrac{ed - bf}{ad - bc} = \dfrac{D_x}{D} \end{array}$$

Notes:

1. If all of the determinants are 0, then the system is dependent.
2. If $D = 0$, but at least one of the other determinants is not 0, then the system is inconsistent.

Examples

1.

$$\begin{cases} 2x + 3y = 0 \\ 4x - 6y = -5 \end{cases}$$

We will solve this system using Cramer's Rule.

This is the augmented matrix we will use:

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & -6 & -5 \end{array}\right]$$

Let's calculate the determinants that we will use the find the value of the variables x and y.

$$D = \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -12 - 12 = -24$$

$$D_x = \begin{vmatrix} 0 & 3 \\ -5 & -6 \end{vmatrix} = 0 - (-15) = 15$$

$$D_y = \begin{vmatrix} 2 & 0 \\ 4 & -5 \end{vmatrix} = -10 - 0 = -10$$

Now we will calculate the variables using the values of the determinants above.

$$x = \frac{D_x}{D} = \frac{15}{-24} = -\frac{15}{24} = -\frac{5}{8}$$

$$y = \frac{D_y}{D} = \frac{-10}{-24} = \frac{10}{24} = \frac{5}{12}$$

So, the solution is $\left(-\frac{5}{8}, \frac{5}{12}\right)$.

2.

$$\begin{cases} 5x + 6y = 12 \\ 10x + 12y = 24 \end{cases}$$

This is the augmented matrix we will use:

$$\left[\begin{array}{cc|c} 5 & 6 & 12 \\ 10 & 12 & 24 \end{array}\right]$$

Now, let's calculate the determinants that we will use the find the value of the variables *x* and *y*.

$$D = \begin{vmatrix} 5 & 6 \\ 10 & 12 \end{vmatrix} = 60 - 60 = 0$$

$$D_x = \begin{vmatrix} 12 & 6 \\ 24 & 12 \end{vmatrix} = 144 - 144 = 0$$

$$D_y = \begin{vmatrix} 5 & 12 \\ 10 & 24 \end{vmatrix} = 120 - 120 = 0$$

Since the values of all of the determinants are 0, then it is a **dependent system**.

3.

$$\begin{cases} 3x + 2y = 11 \\ 6x + 4y = 11 \end{cases}$$

This is the augmented matrix we will use:

$$\left[\begin{array}{cc|c} 3 & 2 & 11 \\ 6 & 4 & 11 \end{array}\right]$$

Now, let's calculate the determinants that we will use the find the value of the variables *x* and *y*.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 12 - 12 = 0$$

$$D_x = \begin{vmatrix} 11 & 2 \\ 11 & 4 \end{vmatrix} = 44 - 22 = 22$$

Since $D = 0$ and at least one of the other determinants is not 0, then the system is **inconsistent**.

Many of the steps we used to solve a system of equations with two equations and two unknowns will be used to solve a system of three equations with three unknowns. We will begin by rewriting the system as an augmented matrix.

Next, we will use several determinants to get values for the variables *x*, *y*, and *z*. We will use an *x*-determinant which will be labeled D_x, a *y*-determinant which will be labeled D_y, a *z*-determinant which will be labeled D_z, and a denominator determinant which will be labeled D.

Then we will use those values in the formulas below to calculate the values of the variables *x*, y, and *z*.

$$x = \frac{D_x}{D}; \quad y = \frac{D_y}{D}; \quad z = \frac{D_z}{D}$$

Because the system that we are solving has 3 equations and 3 unknowns, the determinants that we will be using will be 3×3 determinants which makes the calculations more complicated and time consuming.

4. $$\begin{cases} 3x + 2y - z = -8 \\ 2x - y + 7z = 10 \\ 2x + 2y - 3z = -10 \end{cases}$$

This is the augmented matrix we will use:

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -8 \\ 2 & -1 & 7 & 10 \\ 2 & 2 & -3 & -10 \end{array}\right]$$

Now let's calculate the determinants that we will use the find the value of the variables *x* and *y*.

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 7 \\ 2 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} -1 & 7 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 7 \\ 2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(3-14) - 2(-6-14) - 1(4-(-2))$$

$$= 3(-11) - 2(-20) - 1(6)$$

$$= -33 + 40 - 6$$

$$= 1$$

$$D_x = \begin{vmatrix} -8 & 2 & -1 \\ 10 & -1 & 7 \\ -10 & 2 & -3 \end{vmatrix} = -8\begin{vmatrix} -1 & 7 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 10 & 7 \\ -10 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 10 & -1 \\ -10 & 2 \end{vmatrix}$$

$$= -8(3-14) - 2(-30-(-70)) - 1(20-10)$$

$$= -8(-11) - 2(40) - 1(10)$$

$$= 88 - 80 - 10$$

$$= -2$$

$$D_y = \begin{vmatrix} 3 & -8 & -1 \\ 2 & 10 & 7 \\ 2 & -10 & -3 \end{vmatrix} = 3\begin{vmatrix} 10 & 7 \\ -10 & -3 \end{vmatrix} - (-8)\begin{vmatrix} 2 & 7 \\ 2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 10 \\ 2 & -10 \end{vmatrix}$$

$$= 3(-30-(-70)) + 8(-6-14) - 1(-20-20)$$

$$= 3(40) + 8(-20) - 1(-40)$$

$$= 120 - 160 + 40$$

$$= 0$$

$$D_z = \begin{vmatrix} 3 & 2 & -8 \\ 2 & -1 & 10 \\ 2 & 2 & -10 \end{vmatrix} = 3\begin{vmatrix} -1 & 10 \\ 2 & -10 \end{vmatrix} - 2\begin{vmatrix} 2 & 10 \\ 2 & -10 \end{vmatrix} + (-8)\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(10-20) - 2(-20-20) - 8(4-(-2))$$

$$= 3(-10) - 2(-40) - 8(6)$$

$$= -30 + 80 - 48$$

$$= 2$$

$$x = \frac{D_x}{D} = \frac{-2}{1} = -2$$

$$y = \frac{D_y}{D} = \frac{0}{1} = 1$$

$$z = \frac{D_z}{D} = \frac{2}{1} = 2$$

So, the solution is $\left(-2,\ 1,\ 2\right)$.