8.3 Solving Linear Systems of Equations Using Matrices

As we have already seen, there is more than one way to solve a system of equations. We have already explored the substitution method (the more general method) and the elimination method (a method that works for linear systems). We will now turn our attention to a method that uses "shorthand" for elimination by just using the coefficients rather than the variables from the equations. This method is particularly useful when attempting to solve a linear system of equations with 4 or more variables because it makes the work more compact and less cumbersome. Try to remember that the steps you use for matrices represent the same steps used in elimination and this section will be easier to digest. First, we must introduce the terminology and notation for matrices and then we can use them to solve equations.

Definition: An *m x n* **matrix** is any rectangular array of numbers arranged in *m* rows and *n* columns.

The numbers in the matrix are called **elements**.

$$\begin{bmatrix} 2 & -1 & 3 & 5 \\ 7 & 8 & 19 & -28 \\ -15 & 22 & 81 & 1 \end{bmatrix}$$

In the above matrix, 3 rows and 4 columns are present, so we say this is a *3 x 4* matrix (read as "3 by 4").

In general, $m \times n$ matrices represent expressions.

\# rows # columns

Each row represents an expression while each column represents a variable. For example, the first row of the matrix above represents an expression like $2a - b + 3c + 5d$, if we use a, b, c and d as our variables.

Augmented matrices have a bar and another column. They are used to represent systems of equations as follows:

$$\begin{array}{ccc} x & y & \text{numbers} \\ \multicolumn{3}{c}{\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -1 & 4 \end{array}\right]} \\ & & = \end{array} \quad \text{means} \quad \begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$$

In the augmented matrix above, each column contains the coefficients of a specific variable. For example, the first column contains the coefficients of the variable *x* while the second column contains the coefficients of *y*. The bar represents the equals sign and the numbers (or constants) are on the other side of it. In this way, each row represents an equation.

In the first example that follows, we will solve the system using elimination first and then matrices so that you can see the connection between these processes. Elimination may seem easier to perform, especially on systems that are small (like 2 variables), but performing matrices on small systems like this helps us to get used to (and understand) the process so that we can use it on the larger systems where it is really the most valuable!

Examples

1.

$$\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$$

Elimination Method

$$2x + y = 5$$
$$-2(x - y = 4)$$

$$\begin{aligned} 2x + y &= 5 \\ -2x + 2y &= -8 \\ \hline 3y &= -3 \\ \frac{3y}{3} &= \frac{-3}{3} \\ y &= -1 \end{aligned}$$

substitute in equation

to solve for x.

$$\begin{aligned} 2x + (-1) &= 5 \\ +1 \quad &\ \ +1 \\ \hline 2x \quad &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

The solution is $(3, -1)$.

Now, we will solve the same system of equations using matrices so that we can get used to the notation and the process. Note that documenting each step is important to keep track of what you did and to be able to check your work. You will multiply rows by numbers and add them to other rows to eliminate variables. Note that R_i denotes the ith row in the matrix.

Matrices Method

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -1 & 4 \end{array}\right]$$

$$\begin{array}{llll} -2R_2 + R_1 \rightarrow & -2 & 2 & -8 \\ & +\ 2 & 1 & 5 \\ \hline \text{new } R_2 & 0 & 3 & -3 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

$$R_2 \div 3 \rightarrow \frac{0}{3} \quad \frac{3}{3} \quad \frac{-3}{3}$$

$$\text{new } R_2 \rightarrow 0 \quad 1 \quad -1$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

Rewrite matrix as equations.

$$\begin{cases} 2x + y = 5 \\ \quad\ \ y = -1 \end{cases}$$

Subsititute $y = -1$ in equation to solve for x.

$$2x + (-1) = 5$$
$$2x = 6$$
$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

The notation $-2R_2 + R_1$ means the second row was multiplied by -2 and added to the first row. You can see the computation next to the notation. We replace row 2 with our result (new equation).

The solution is $(3, -1)$.

How is this helpful?

If you have 4 equations in 4 variables, you can use matrices to get:

0's on top not necessary since we can back substitute

Row Reduced Echelon Form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 1 & 52 \end{array}\right] \begin{array}{l} x = 5 \\ y = -8 \\ z = 13 \\ r = 52 \end{array}$$

zeros below are necessary (eliminates variables)

What are we allowed to do? <u>Row Operations</u>

I. Switch rows
II. Multiply (or divide) rows by numbers (other than 0)
III. Add rows together

2.

$$\begin{cases} x + y + z = 7 \\ x + 2y + z = 8 \\ x + y + 2z = 10 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 2 & 10 \end{array}\right]$$

Get 0's beneath diagonal (one column at a time).

Using matrices:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 2 & 10 \end{array}\right] R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 2 & 10 \end{array}\right]$$

Replacing R_2, so copy down R_1 and R_3.

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow{R_3 \div (-1)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

replace row 3
(subtract R_3 from R_1)

replace row 3
(divide R_3 by -1)

Now, rewrite the last matrix as equations:

$$\begin{cases} x + y + z = 7 \\ \quad -y = -1 \quad \Rightarrow y = 1 \\ \quad\quad z = 3 \end{cases}$$

Next, substitute values for y and z into the first equation to solve for x.

$$x + 1 + 3 = 7$$
$$x + 4 = 7$$
$$x = 3$$

The solution is $(3, 1, 3)$.

Make sure that you check your solution!

3. $\begin{cases} 2x + y - 3z = -1 \\ 3x - 2y - z = -5 \\ x - 3y - 2z = -12 \end{cases}$

Switch R_1 and R_3

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & -1 \\ 3 & -2 & -1 & -5 \\ 1 & -3 & -2 & -12 \end{array}\right] R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & -2 & -12 \\ 3 & -2 & -1 & -5 \\ 2 & 1 & -3 & -1 \end{array}\right]$$

$$\begin{array}{r} -3R_1 + R_2 \rightarrow \\ \\ \text{new } R_2 \rightarrow \end{array} \begin{array}{ccc|c} -3 & 9 & 6 & 36 \\ 3 & -2 & -1 & -5 \\ \hline 0 & 7 & 5 & 31 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -12 \\ 0 & 7 & 5 & 31 \\ 2 & 1 & -3 & -1 \end{array}\right]$$

$$\begin{array}{r} -2R_1 + R_3 \rightarrow \\ \\ \text{new } R_3 \rightarrow \end{array} \begin{array}{ccc|c} -2 & 6 & 4 & 24 \\ 2 & 1 & -3 & -1 \\ \hline 0 & 7 & 1 & 23 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -12 \\ 0 & 7 & 5 & 31 \\ 0 & 7 & 1 & 23 \end{array}\right]$$

$$\begin{array}{r} R_2 - R_3 \rightarrow \\ \\ \text{new } R_3 \rightarrow \end{array} \begin{array}{ccc|c} 0 & 7 & 5 & 31 \\ 0 & -7 & -1 & -23 \\ \hline 0 & 0 & 4 & 8 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -12 \\ 0 & 7 & 5 & 31 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$R_3 \div 4 \rightarrow \quad \frac{0}{4} \quad \frac{0}{4} \quad \frac{4}{4} \quad \Big| \frac{8}{4}$$

$$\text{new } R_3 \rightarrow \quad 0 \quad 0 \quad 1 \quad | \; 2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -12 \\ 0 & 7 & 5 & 31 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Write matrix as equations.

$$\begin{cases} x - 3y - 2z = -12 \\ 7y + 5z = 31 \\ z = 2 \end{cases}$$

Back substitute to solve.

$$7y + 5(2) = 31$$
$$7y + 10 = 31$$
$$7y = 21$$
$$\frac{7y}{7} = \frac{21}{7}$$
$$y = 3$$

Back substitute again to solve.

$$x - 3(3) - 2(2) = -12$$
$$x - 9 - 4 = -12$$
$$x - 13 = -12$$
$$x = 1$$

The solution is $(1,\ 3,\ 2)$.

4.

$$\begin{cases} x+3y-z=3 \\ 2x+y+7z=10 \\ 4x-3y+2z=3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 2 & 1 & 7 & 10 \\ 4 & -3 & 2 & 3 \end{array}\right]$$

$$\begin{array}{lrrrr} -2R_1+R_2 \to & -2 & -6 & 2 & -6 \\ & 2 & 1 & 7 & 10 \\ \hline \text{new } R_2 \to & 0 & -5 & 9 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 9 & 4 \\ 4 & -3 & 2 & 3 \end{array}\right]$$

$$\begin{array}{lrrrr} -4R_1+R_3 \to & -4 & -12 & 4 & -12 \\ & 4 & -3 & 2 & 3 \\ \hline \text{new } R_3 \to & 0 & -15 & 6 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 9 & 4 \\ 0 & -15 & 6 & -9 \end{array}\right]$$

$$R_3 \div (-3) \to \quad \frac{0}{-3} \quad \frac{-15}{-3} \quad \frac{6}{-3} \quad \frac{-9}{-3}$$

$$\text{new } R_3 \quad 0 \quad 5 \quad -2 \quad 3$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 9 & 4 \\ 0 & 5 & -2 & 3 \end{array}\right]$$

$$\begin{array}{lcccc} R_2 + R_3 \rightarrow & 0 & -5 & 9 & 4 \\ & 0 & 5 & -2 & 3 \\ \hline \text{new } R_3 \rightarrow & 0 & 0 & 7 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 9 & 4 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

$$R_3 \div 7 \rightarrow \quad \frac{0}{7} \quad \frac{0}{7} \quad \frac{7}{7} \quad \frac{7}{7}$$

$$\text{new } R_3 \rightarrow \quad 0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -5 & 9 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Write matrix as equations.

$$\begin{cases} x + 3y - z = 3 \\ -5y + 9z = 4 \\ z = 1 \end{cases}$$

Now, back substitute to solve.

$$\begin{aligned} -5y + 9(1) &= 4 \\ -5y + 9 &= 4 \\ -5y &= -5 \\ \frac{-5y}{-5} &= \frac{-5}{-5} \\ y &= 1 \end{aligned}$$

Back substitute again.

$$\begin{aligned} x + 3(1) - 1 &= 3 \\ x + 3 - 1 &= 3 \\ x + 2 &= 3 \\ x &= 1 \end{aligned}$$

The solution is $(1, 1, 1)$.