8.2 Systems of Linear Equations in Three Variables

Three variable systems are solved the same way that two variable systems are solved, but the process has to be repeated. When we have three linear equations in three variables, we can use elimination on pairs of equations to eliminate the same variable and then put our resulting two variable equations together to eliminate another variable. The algebraic and geometric meanings are analogous to those from the two variable systems, except that we have three dimensions of space with three variables.

Algebraic Meaning

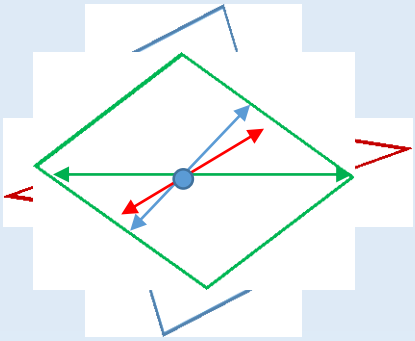
A solution to a system of three linear equations in three variables must satisfy all three equations. For example, the solution (2,4,6) satisfies the system

$$\begin{cases} 3x - y + 2z = 14 \\ x + y - z = 0 \\ 2x - y + 3z = 18 \end{cases}$$

because if you plug the point into any of the equations, it makes the equation true:

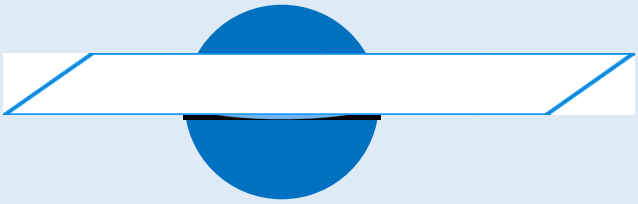
$$\begin{cases} 3(2) - 4 + 2(6) = 14 \\ 2 + 4 - 6 = 0 \\ 2(2) - 4 + 3(6) = 18 \end{cases}$$

Geometric Meaning



Each linear equation in three variables can be represented by a *plane,* a two dimensional flat surface (like a piece of paper that extends forever in every direction). The intersection of two planes is a line, when it exists. When a third plane is added, the intersection of all three is often a point. (Sometimes, you have planes that do not all intersect).

The same idea applies when we have non-linear equations, but our graphs will not necessarily be planes. We may have more than one intersection in this case as well. The solutions can be two dimensional surfaces. This topic is beyond the scope of this course, but it helps to see these connections.



This picture is a plane intersecting a sphere, which gives us a circle.

<u>Examples</u>

Solve each system of equations.

1. $\begin{cases} 3x - y + 2z = 14 \\ x + y - z = 0 \\ 2x - y + 3z = 18 \end{cases}$

In order to use elimination on this system, we will first choose a variable to eliminate, for example y. Then we will pair up our equations twice in any way we choose (as long as the pairs are different) to eliminate that variable. For example, if we choose the first two and the last two as our pairs, we get:

$$\begin{array}{r} 3x - y + 2z = 14 \\ x + y - z = 0 \\ \hline 4x \qquad + z = 14 \end{array} \qquad\qquad \begin{array}{r} x + y - z = 0 \\ 2x - y + 3z = 18 \\ \hline 3x \qquad + 2z = 18 \end{array}$$

We have eliminated the same variable in both sets, so the same variables are left over. This means we can put our two resulting

equations together and eliminate another variable. We now have a system of two equations in two variables.

$$\begin{cases} 4x + z = 14 \\ 3x + 2z = 18 \end{cases}$$

We will eliminate z by multiplying the first equation by -2 and adding it to the second equation as follows:

$$\begin{aligned} -2(4x + z) &= 14(-2) \\ 3x + 2z &= 18 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -8x - 2z &= -28 \\ 3x + 2z &= 18 \\ \hline -5x \qquad &= -10 \end{aligned}$$

$$\Rightarrow \quad x = 2$$

Now that we have x, we can plug this value back into either equation to get the other variable.

$$4x + z = 14$$

$$\Rightarrow \quad 4(2) + z = 14$$

$$\Rightarrow \quad \begin{aligned} 8 + z &= 14 \\ -8 \qquad &- 8 \\ \hline \end{aligned}$$

$$\Rightarrow \quad z = 6$$

Now that we have both x and z, we can plug them back into any of the original three equations to get the last variable, y.

$$x + y - z = 0$$

$$\Rightarrow \quad 2 + y - 6 = 0$$

$$\Rightarrow \quad \begin{aligned} y - 4 &= 0 \\ +4 \quad &+ 4 \end{aligned}$$

$$\Rightarrow \quad y = 4$$

Therefore, the solution is the point $(2,4,6)$.

2. $\begin{cases} 2x - 4y + z = 5 \\ 3x + y - z = 4 \\ 2x - 3y + 5z = -10 \end{cases}$

We will first choose a variable to eliminate. It doesn't matter which one, but z looks like the simplest choice. If we choose the first two and the last two as our pairs, we get:

$$\begin{array}{r} 2x - 4y + z = 5 \\ 3x + y - z = 4 \\ \hline 5x - 3y \quad = 9 \end{array} \qquad \begin{array}{r} 5(3x + y - z) = 4(5) \\ 2x - 3y + 5z = -10 \end{array}$$

$$\begin{array}{r} 15x + 5y - 5z = 20 \\ 2x - 3y + 5z = -10 \\ \hline 17x + 2y \qquad = 10 \end{array}$$

We have eliminated the same variable in both sets, so the same variables are left over. This means we can put our two resulting equations together and eliminate another variable. We now have a system of two equations in two variables.

$$\begin{cases} 5x - 3y = 9 \\ 17x + 2y = 10 \end{cases}$$

We will eliminate y by multiplying the first equation by 2 and the second equation by 3 and adding them.

$$\begin{array}{r} 2(5x - 3y) = 9(2) \\ 3(17x + 2y) = 10(3) \end{array} \quad \Rightarrow \quad \begin{array}{r} 10x - 6y = 18 \\ 51x + 6y = 30 \\ \hline 61x \qquad = 48 \end{array}$$

$$\Rightarrow \quad x = \frac{48}{61}$$

Now that we have x, we can plug this value back into either equation to get the other variable (or use elimination again).

$$5x - 3y = 9$$

$$\Rightarrow \quad 5\left(\frac{48}{61}\right) - 3y = 9$$

$$\Rightarrow \quad \frac{240}{61} - 3y = 9$$

$$-\frac{240}{61} \qquad -\frac{240}{61}$$

$$\Rightarrow \quad -3y = \frac{309}{61}$$

$$9 - \frac{240}{61}$$
$$= \frac{9}{1} - \frac{240}{61}$$
$$= \frac{9}{1} \cdot \frac{61}{61} - \frac{240}{61}$$
$$= \frac{549}{61} - \frac{240}{61} = \frac{309}{61}$$

$$\Rightarrow \quad y = -\frac{309}{3 \cdot 61} = -\frac{103}{61}$$

Now that we have both x and y, we can plug them back into any of the original three equations to get the last variable, z.

$$3x + y - z = 4$$

$$\Rightarrow \quad 3\left(\frac{48}{61}\right) + \left(-\frac{103}{61}\right) - z = 4$$

$$\Rightarrow \quad \frac{144}{61} - \frac{103}{61} - z = 4$$

$$\Rightarrow \quad \frac{41}{61} - z = 4$$

$$-\frac{41}{61} \qquad -\frac{41}{61}$$

$$\Rightarrow \quad -z = \frac{203}{61}$$

$$\Rightarrow \quad z = -\frac{203}{61}$$

$$4 - \frac{41}{61}$$
$$= \frac{4}{1} - \frac{41}{61}$$
$$= \frac{4}{1} \cdot \frac{61}{61} - \frac{41}{61}$$
$$= \frac{244}{61} - \frac{41}{61} = \frac{203}{61}$$

Therefore, the solution is the point $\left(\frac{48}{61}, -\frac{103}{61}, -\frac{203}{61}\right)$.

3. $\begin{cases} 2x + 3y - 2z = -5 \\ 5x - 6\,y + z = 19 \\ 4x - 2y - 6z = -8 \end{cases}$

We will first choose a variable to eliminate. The variable z looks like the simplest choice. If we choose the first two and the last two as our pairs, we get:

$$\begin{array}{c} 2x + 3y - 2z = -5 \\ 2(5x - 6y + z) = 19(2) \end{array} \qquad \begin{array}{c} 6(5x - 6y + z) = 19(6) \\ 4x - 2y - 6z = -8 \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{array}{r} 2x + 3y - 2z = -5 \\ 10x - 12y + 2z = 38 \\ \hline 12x - 9y \qquad = 33 \end{array} \qquad \begin{array}{r} 30x - 36y + 6z = 114 \\ 4x - 2y - 6z = -8 \\ \hline 34x - 38y \qquad = 106 \end{array}$$

Notice that each of these equations can be reduced by dividing both sides by the same number. Dividing the first equation by 3 yields $4x - 3y = 11$. Dividing the second equation by 2 gives us $17x - 19y = 53$. Whenever it is possible to reduce equations it is a good idea because it makes the numbers smaller and working with smaller numbers is simpler than working with larger numbers. We now have a system of two equations in two variables.

$$\begin{cases} 4x - 3y = 11 \\ 17x - 19y = 53 \end{cases}$$

We will eliminate x by multiplying the first equation by 17 and the second equation by -4 and adding them.

$$\begin{aligned} 17(4x-3y) &= 11(17) \\ -4(17x-19y) &= 10(-4) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 68x-51y &= 187 \\ -68x+76y &= -212 \\ \hline 25y &= -25 \end{aligned}$$

$$\Rightarrow \quad y = -1$$

Now that we have y, we can plug this value back into either equation to get the other variable (or use elimination again).

$$4x - 3y = 11$$

$$\Rightarrow \quad 4x - 3(-1) = 11$$

$$\Rightarrow \quad 4x + 3 = 11$$

$$\begin{aligned} -3 \quad &-3 \\ \hline 4x \quad &= 8 \end{aligned}$$

$$\Rightarrow \quad x = 2$$

Now that we have both x and y, we can plug them back into any of the original three equations to get the last variable, z.

$$5x - 6y + z = 19$$

$$\Rightarrow \quad 5(2) - 6(-1) + z = 19$$

$$\Rightarrow \quad 10 + 6 + z = 19$$

$$\Rightarrow \quad \begin{aligned} 16 + z &= 19 \\ -16 \quad &-16 \\ \hline z &= 3 \end{aligned}$$

Therefore, the solution is the point $(2, -1, 3)$.

Don't forget that you have choices when you are solving these systems. You could eliminate y in the beginning, or x, and the equations might be nicer to work with. Try this problem a different way and see that you get the same answer.

4. $\begin{cases} x - y = 3 \\ 2x - y + z = 1 \\ x + z = -2 \end{cases}$

There is more than one way to begin here. Since we have equations with only two variables, we could pair up two of them to get an equation that matches the variables in the other one, or we could proceed by eliminating the common variable, *x*. You could also use substitution if you solve the first and last in terms of x and plug your results into the second equation. We choose to eliminate the common variable, x:

$$\begin{array}{r} -2(x-y) \quad = 3(-2) \\ 2x - y + z = 1 \end{array} \qquad \begin{array}{r} 2x - y + z = 1 \\ -2(x \quad + z) = -2(-2) \end{array}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{r} -2x + 2y \quad = -6 \\ 2x - y + z = 1 \\ \hline y + z = -5 \end{array} \qquad \begin{array}{r} 2x - y + z = 1 \\ -2x \quad - 2z = 4 \\ \hline -y - z = 5 \end{array}$$

You may notice that these equations are really the same, since one is a multiple of the other. If you add them together, you get a true statement, but the variables drop out, which means you have a dependent system.

$$\begin{array}{r} y + z = -5 \\ -y - z = 5 \\ \hline 0 = 0 \end{array}$$

Even though we have a dependent system, you may notice that the planes are all different (none is a multiple of any of the others). The picture for this situation is three planes that all cross in a line, which means we do indeed have infinitely many solutions.

Similarly, you may have a situation where the planes do not all three have an intersection that they share. Sometimes, two or more of the planes are parallel, but you can also have a situation where each pair has an intersection, but not all three. Algebraically, the variables drop out and you get an untrue statement, just as in the two variable systems. In that case, you have an inconsistent system.

Applications

The same technique that we used to set up two variable systems applies to three variable systems as well. The main difference is that you end up with three equations instead of two.

Examples

Set up a system of equations for each of the following applications.

1. (Numbers)

   The sum of three numbers is 5. The first number less the second number increased by the third number is 1. The first decreased by the third is three more than the second. Find the numbers.

The question here is asking for three numbers, and that means we should end up with two equations. Let's name our variables and then we will write down three equations in those variables.

Let x=first number
y=second number
z=third number

"The sum of three numbers is 5" translates to:

$$x + y + z = 5$$

"The first number less the second number increased by the third number is 1" translates to:

$$x - y + z = 1$$

"The first decreased by the third is three more than the second" translates to:

$$x - z = y + 3$$

So our system is $\begin{cases} x + y + z = 5 \\ x - y + z = 1 \\ x - z = y + 3 \end{cases}$

2. (Geometry)

In triangle ABC, the measure of angle B is five times that of angle A. The measure of angle C is 60° more than that of angle A. Find the angle measures.

We can name our three angles A, B, and C.

From the first sentence, we get the equation:

$$B = 5A$$

From the second sentence, we get the equation:

$$C = A + 60$$

But we are out of sentences!

Sometimes, you have to bring prior knowledge to a problem. In this case, we know that the sum of the interior angles of any triangle equals 180, so we can use that information to write down another equation:

$$A + B + C = 180$$

So our system is $\begin{cases} A + B + C = 180 \\ B = 5A \\ C = A + 60 \end{cases}$

3. (Investments)

A total of $80,000 was invested in three mutual funds, one at 10% interest, the second at 6% interest and the third at 15% interest. If the annual income from these combined investments is $8850 and the earnings from the first fund were $750 more than the earnings from the third fund, how much was invested at each rate?

The questions asks “how much was invested at each rate?”, so we know we have three variables since there are three rates. We need to name them:

Let x =amount invested at 10%
y = amount invested at 6%
z = amount invested at 15%

The amounts invested at each rate should add up to the total amount invested:

$$x + y + z = 80{,}000$$

The earnings from each investment should add up to the total earnings:

$$.1x + .06y + .15z = 8850$$

Translating the phrase “the earnings from the first fund were $750 more than the earnings from the third fund”, we get the third equation:

$$.1x = .15z + 750$$

So our system is $\begin{cases} x + y + z = 80{,}000 \\ .1x + .06y + .15z = 8850 \\ .1x = .15z + 750 \end{cases}$