# 8.1 Systems of Linear Equations in Two Variables

We now return to the major concept of solving equations, but this time we are solving "sets" of equations called systems of equations. This means that the solutions have to satisfy ALL of the equations in the set. Our main focus will be on linear systems, but at higher levels of math, you will explore non-linear systems. Now that we have done a bit of graphing, we can talk about the geometric meaning of these solutions as well.

## Algebraic Meaning

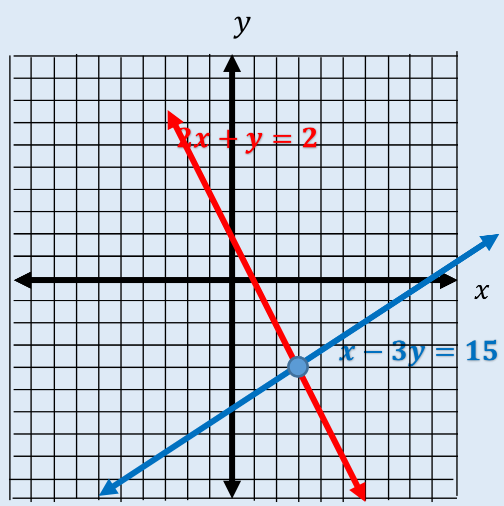
In order for a point to be a solution to a system of equations, it must satisfy both equations. For example, the point $(3, -4)$ is a solution of the linear system

$$\begin{cases} x - 3y = 15 \\ 2x + y = 2 \end{cases}$$

Because if you plug the point into either equation, it makes the equation true:

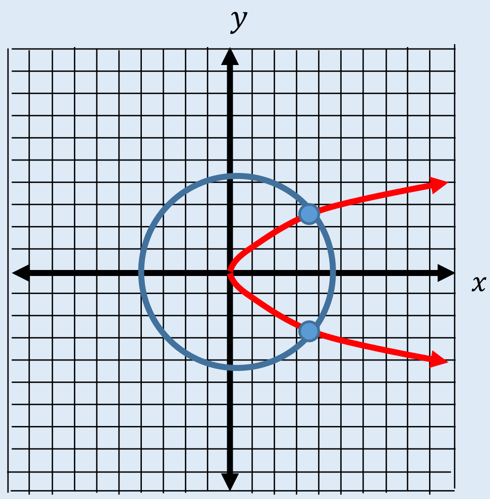
$$\begin{cases} 3 - 3(-4) = 15 \\ 2(3) + (-4) = 2 \end{cases}$$

## Geometric Meaning




The point $(3, -4)$ lies on both lines. It is the **intersection** of their graphs! The geometric meaning of a solution is the place where the graphs intersect.

The same idea applies when we have non-linear equations, but our graphs will not necessarily be lines. We may have more than one intersection in this case. The solutions are still the points of intersection and those points will satisfy both equations.




In order to solve systems of equations algebraically, the most general method is to use **substitution**, where you solve one of the equations in one of the variables and plug it into the other equation. This method will work on linear systems, but also on most non-linear systems as well. There is another method called **elimination** that will work for linear systems, but it is not generally useful for non-linear systems. To use elimination, you will multiply your equations by numbers so that when you add the equations together, one of the variables will drop out. We will use elimination for the majority of our examples because we are working with linear systems and this method is utilized in developing the operations for matrices (in the next section).

**Examples**

**Solve each system of equations.**

1. $\begin{cases} x + 3y = 7 \\ 2x - 5y = -8 \end{cases}$

For the first example, we will use the method of substitution so that both methods are presented here. To use substitution, choose an equation and a variable to solve for. It is nice when you have a variable without a coefficient because those are very easy to solve for. Here, we will choose the first equation and we will solve it for the variable $x$:

$$x + 3y = 7$$

$$\Rightarrow \quad x = 7 - 3y$$

Now, we will replace $x$ in the other equation with $7 - 3y$.

$$2x - 5y = -8$$

$$\Rightarrow \quad 2(7 - 3y) - 5y = -8$$

We can now solve this equation for $y$:

$$\Rightarrow \quad 14 - 6y - 5y = -8$$

$$\Rightarrow \quad 14 - 11y = -8$$

$$\underline{-14 \qquad\qquad -14}$$

$$\Rightarrow \quad -11y = -22$$

$$\Rightarrow \quad y = 2$$

So, now we have the $y$ value for the point of intersection, but we still need to find the $x$ value. Since our solution will satisfy either equation, we can plug our $y$ value back into either equation to find $x$. It does not matter which equation we choose – we will get the same answer! We are choosing the first equation, since one form of it is already solved for $x$.

$$x = 7 - 3y$$

$$\Rightarrow \quad x = 7 - 3(2)$$

$$\Rightarrow \quad x = 1$$

Therefore, the solution is the point $(1,2)$.

2. $\begin{cases} x + 3y = 7 \\ 2x - 5y = -8 \end{cases}$

We will consider the same example we just did but now we will use the method of elimination. To use the method of elimination, we will first decide which variable we wish to eliminate first.

We will eliminate $x$ by multiplying the first equation by -2 and adding it to the second equation as follows:

$$\begin{aligned} -2(x+3y) &= 7(-2) \\ 2x - 5y &= -8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -2x - 6y &= -14 \\ \underline{2x - 5y} &\underline{= -8} \\ -11y &= -22 \end{aligned}$$

$$\Rightarrow \quad y = 2$$

Make sure you multiply **both** sides of the equation(s) by the number so that you do not change your equation(s).

Now that we have $y$, we can either plug this value back into either equation (like we did in example 1) or we can use elimination again on our system to eliminate the other variable, $y$. We will choose to perform elimination again to show the process once more. Begin with your original system:

$$\begin{aligned} x + 3y &= 7 \\ 2x - 5y &= -8 \end{aligned}$$

To eliminate the variable $y$, we will multiply the first equation by 5 and the second equation by 3:

$$\begin{aligned} 5(x+3y) &= 7(5) \\ 3(2x-5y) &= -8(3) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 5x + 15y &= 35 \\ \underline{6x - 15y} &\underline{= -24} \\ 11x &= 11 \end{aligned}$$

$$\Rightarrow \quad x = 1$$

Therefore, the solution is the point $(1,2)$.

Sometimes, strange things happen when we are solving a system of equations….

3. $\begin{cases} x + 2y = 3 \\ 2x + 4y = 6 \end{cases}$

You might notice that one of these equations is a multiple of the other equation. When that happens, we actually have two of the same equation. That means that these two equations represent the same line. That means that every point on the line is a solution to this system! It seems redundant, but if we think of them as two lines, then the system really does have infinitely many solutions. We will talk about how to write that down after discussing the algebra. Let's see what happens.

To eliminate the variable $x$, we will multiply the first equation by $-2$:

$$\begin{aligned} -2(x+2y) &= 3(-2) \\ 2x + 4y &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -2x - 4y &= -6 \\ 2x + 4y &= 6 \\ \hline 0 &= 0 \end{aligned}$$

Notice that all of the variables dropped out and you are left with a true statement. Whenever this happens, you are in this type of situation where two of the equations represent the same thing.

Therefore, the solution is the set of points that describes the line in either form: $\{(x, y): x + 2y = 3\}$ or you could write it in a shorter form if you solve for $y$ and replace that variable in terms of $x$ in the point $(x, y)$:

$$\left(x, -\frac{1}{2}x + \frac{3}{2}\right).$$

We call this type of system a **dependent** system.

4. $\begin{cases} x + 2y = 3 \\ 2x + 4y = 5 \end{cases}$

If we change the example we just did very slightly, you will notice that one side of the equation is a multiple of one side of the other equation, but the other sides are not related that way. If you were to solve each of these equations for the variable $y$, you would see that these two lines have the same slope, but different $y$-intercepts. This means that they are parallel lines! We know that parallel lines do not cross each other, so there is no solution to this system of equations. Let's see what happens algebraically.

To eliminate the variable $x$, we will multiply the first equation by $-2$:

$$\begin{array}{c} -2(x + 2y) = 3(-2) \\ 2x + 4y = 5 \end{array} \quad \Rightarrow \quad \begin{array}{r} -2x - 4y = -6 \\ \underline{2x + 4y = 5} \\ 0 = -1 \end{array}$$

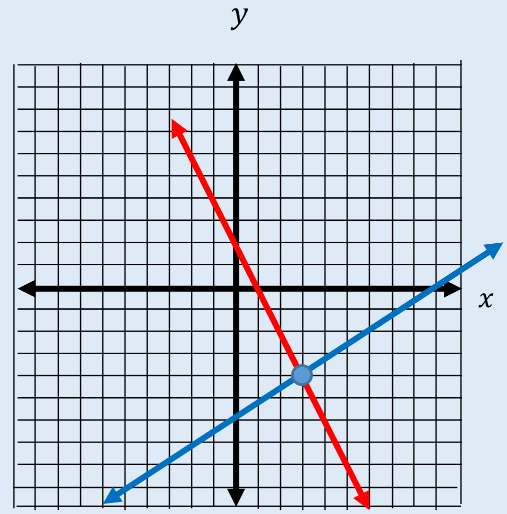
Notice that all of the variables dropped out and you are left with an untrue statement. Whenever this happens, you are in this type of situation where the lines are parallel and do not cross.

Therefore, there is no solution. We call this type of system an **inconsistent** system.

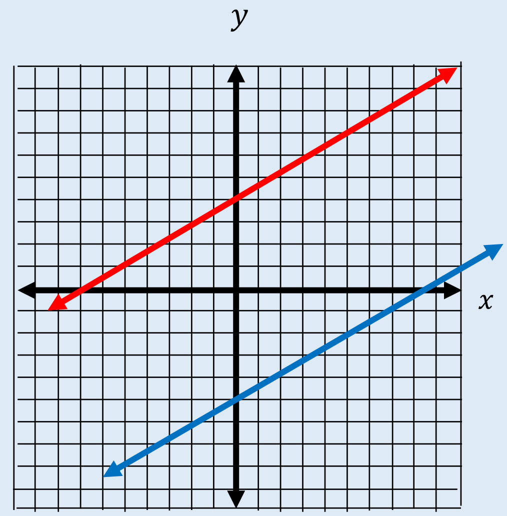
To summarize these new words, a **dependent** system is one in which the equations represent the same line and an **independent** system is one in which the lines are different. An **inconsistent** system is one in which there is no solution and a **consistent** system is a system that has at least one solution. Notice that a dependent system is consistent, since there are solutions.

## <u>Summary</u>

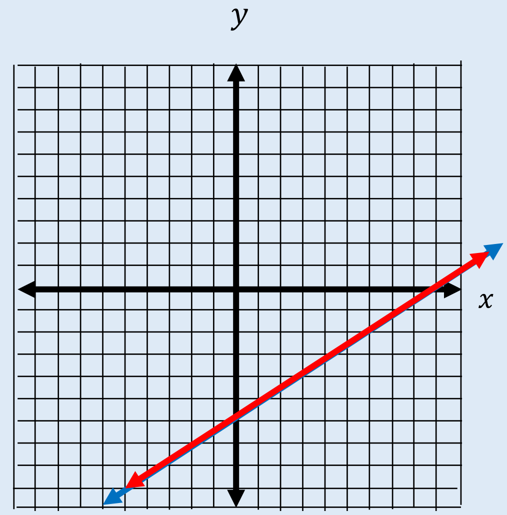
This system is **consistent** and **independent** since there is at least one solution and they are different lines.




This system is **inconsistent** and **independent** since there is no solution and they are different lines. (Parallel lines)




This system is **consistent** and **dependent** since there is at least one solution and they are the same line.



If you encounter a problem that has fractions or decimals in either equation, an easy approach is to clear the fractions and/or decimals to get a new system with integer coefficients. Then simply use elimination on the nicer looking system.

5. $$\begin{cases} \frac{1}{2}x - \frac{1}{4}y = \frac{1}{16} \\ 0.7y = 1 - 0.3x \end{cases}$$

To clear the fractions in the first equation, multiply both sides by the common denominator, 16:

$$16\left(\frac{1}{2}x - \frac{1}{4}y\right) = \frac{1}{16} \cdot 16$$

$$\Rightarrow \qquad 8x - 4y = 1$$

This is the same equation, just in a different form.

To clear the decimals in the second equation, multiply both sides by 10:

$$10(0.7y) = (1 - 0.3x)10$$

$$\Rightarrow \qquad 7y = 10 - 3x$$

We still need to put this one in standard form so that we can line up the variables to do elimination. Adding $3x$ to both sides, we get:

$$\Rightarrow \qquad 3x + 7y = 10$$

Now, let’s put these two equations together and use elimination.

$$\begin{cases} 8x - 4y = 1 \\ 3x + 7y = 10 \end{cases}$$

One way we can eliminate the variable $x$ is to multiply the first equation by $-3$ and the second equation by 8:

$$\begin{aligned} -3(8x-4y) &= 1(-3) \\ 8(3x+7y) &= 10(8) \end{aligned} \quad \Rightarrow \quad \begin{aligned} -24x+12y &= -3 \\ 24x+56y &= 80 \end{aligned}$$

$$68y = 77$$

$$\Rightarrow \quad y = \frac{77}{68}$$

Now that we have $y$, we can either plug this value back into either equation or we can use elimination again on our system , but it seems much easier to just use elimination again.

$$\begin{aligned} 8x-4y &= 1 \\ 3x+7y &= 10 \end{aligned}$$

To eliminate the variable $y$, we will multiply the first equation by 7 and the second equation by 4:

$$\begin{aligned} 7(8x-4y) &= 1(7) \\ 4(3x+7y) &= 10(4) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 56x-28y &= 7 \\ 12x+28y &= 40 \end{aligned}$$

$$68x = 47$$

$$\Rightarrow \quad x = \frac{47}{68}$$

Therefore, the solution is the point $\left(\frac{47}{68}, \frac{77}{68}\right)$ or in decimal form, approximately $(0.7, 1.1)$.

There are many applications that involve solving systems of equations. We are limited to discuss linear systems at this time, but you will encounter more general systems as you move through your college courses. In economics, you will study supply and demand curves for a given commodity and you will find their intersection to be the market price. Lines are used to approximate those supply and demand curves most of the time. In linear programming applications, business students test intersection points of multiple lines to maximize profit. We will look at some applications next and set up the systems for each situation. We will not solve them, however, as the goal is to be able to translate the words into math.

## Applications

When dealing with application problems, recall that there is a way to approach them that helps to break the problem into pieces.

1. Read through the problem once.
2. Go straight to the question.
3. Name the variables by noting what the question is asking for.
4. Read through the problem one more time, sentence by sentence and translate into equations. (You should arrive at the same number of equations as you have variables).

**Examples**

**Set up a system of equations for each of the following applications.**

1. (Geometry)

   The perimeter of a rectangle is 80 inches. The length is 4 inches less than 3 times the width. What are the dimensions of the rectangle?

   The question here is asking for the dimensions of the rectangle, so we know we are looking for two things: length and width. That means we should end up with two equations. Let’s name our variables and then we will write down two equations in those variables.

   Let $l$=length
   And $w$=width

Now reading through the problem, we see that the perimeter is 80, so we need to write down perimeter in terms of our variables $l$ and $w$. Even if you don't recall the formula $P = 2l + 2w$, you can quickly get the formula by drawing a picture and adding up the sides of the rectangle:


$l + w + l + w = 2l + 2w$

So we have our first equation: $2l + 2w = 80$

To get the second equation, keep reading.

"The length is 4 inches less than 3 times the width."

You can translate this sentence directly since "is" means "equals" and "less than" means to subtract from something, etc.

"The length is" means "$l =$"

Now tack on the phrase "is 4 inches less than" to get

$$l = \text{something} - 4$$

That something is 3 times the width, giving us

$$l = 3w - 4$$

So our system is $\begin{cases} 2l + 2w = 80 \\ l = 3w - 4 \end{cases}$

2. (Investments)

A total of \$5000 was invested in two mutual funds, one at 9% interest and the other at 7% interest. If the annual income from these combined investments is \$950, how much was invested at each rate?

The questions asks “how much was invested at each rate?”, so we know we have two variables since there are two rates. We need to name them:

Let $x =$amount invested at 9%
And $y =$ amount invested at 7%

This type of problem is modelled with an amount equation and a value equation. The amounts invested at each rate should add up to the total amount invested:

$$x + y = 5000$$

The earnings from each investment should add up to the total earnings:

$$.09x + .07y = 950$$

So our system is $\begin{cases} x + y = 5000 \\ .09x + .07y = 950 \end{cases}$

3. (Mixtures)

If gummy bears cost \$3.50 per pound and jelly beans cost \$5.50 per pound, how many pounds of each candy must be mixed to obtain 60 pounds of candy which costs 4.00 per pound?

The question is asking us "how many pounds of each candy…" Again, we know we have two types of candy, so we have two variables and we should end up with two equations. Let's name our variables.

Let $g$ =gummy bears
And $j$ =jelly beans

Mixture problems are also modeled with an amount equation and a value equation. The amounts (number of pounds) of each type of candy should add up to the amount (number of pounds) of the mixture:

$$g + j = 60$$

The value of each type of candy should add up to the value of the mixture:

$$3.5g + 5.5j = 4(60)$$

To get the value of the mixture, you need to multiply the number of pounds by the price per pound.

So our system is $\begin{cases} g + j = 60 \\ 3.5g + 5.5j = 240 \end{cases}$