

7.5 The Binomial Theorem

The Binomial Theorem gives us a formula to put a binomial to a power without actually multiplying it out. We will see that Pascal's Triangle will also give us a way to do this. We will learn some new notation as well as a new operation called "factorial" and denoted with an exclamation point ($n!$) in this lesson as well. We begin by considering some powers of a generic binomial $a + b$:

$$(a + b)^0 = 1$$

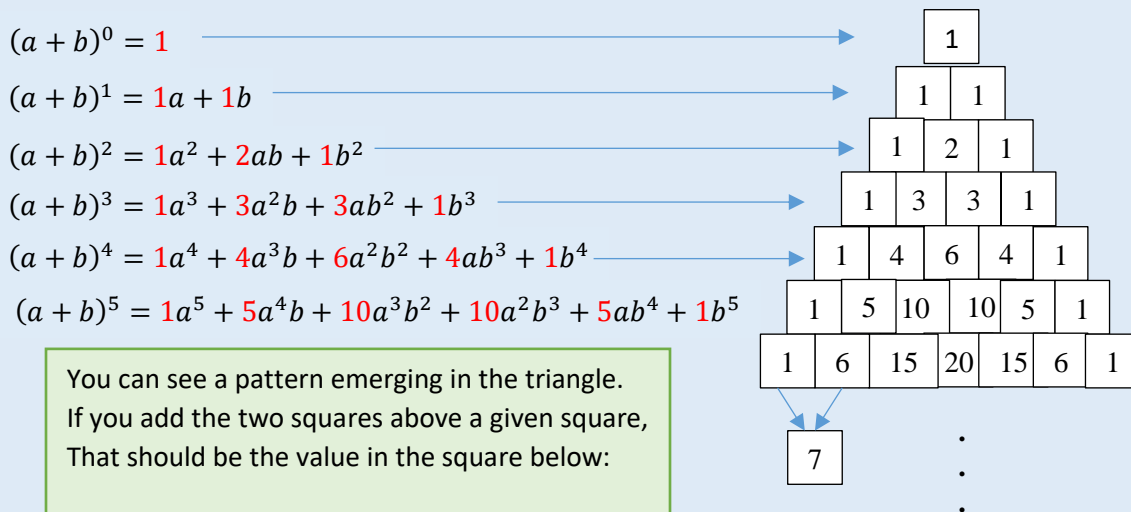
$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

After this, it becomes pretty tedious to multiply out $(a + b)^n$.

If we consider the coefficients, we can find a pattern that will help us predict the coefficients for higher powers of $a + b$. This pattern can be found in Pascal's Triangle as follows:



We can see that the n^{th} row of the triangle is made up of the coefficients of $(a + b)^{n-1}$. For example, the first row represents $(a + b)^0$, the second row represents the coefficients of $(a + b)^1$, the third row represents the coefficients of $(a + b)^2$, and so on. Notice that there is also a pattern with the powers on the variables of each term. When expanding $(a + b)^n$, the powers on a start high (at n) and go down (to 0), whereas the powers on b start low (at 0) and go up (to n). This makes it easy to write down $(a + b)^n$ for any n . First write down the powers of a , then write down the powers of b , and then fill in the coefficients from the triangle. You will also notice that the sum of those powers on a and b in each term is always n .

For example:

To write down $(a + b)^6$, the powers of a go from 6 to 0 as follows:

$$a^6 + a^5 + a^4 + a^3 + a^2 + a^1 + a^0$$

Now, fill in the powers of b , which go from 0 to 6:

$$a^6b^0 + a^5b^1 + a^4b^2 + a^3b^3 + a^2b^4 + a^1b^5 + a^0b^6$$

Now, go to the 7th row in Pascal's Triangle to get the coefficients:

$$1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$$

Finally, simplify:

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + b^6$$

Notice that for each term, the powers add to 6.

To use this method, you will probably want to make a copy of Pascal's Triangle (as much of it as you think you will need) and keep it handy while expanding your binomials. You can always quickly recreate it using the pattern, but it can be tedious if you need the 15th row or something like that.

What happens if we have a binomial with more complicated terms that simply a and b ? Not much will change except that you will need to use a substitution for a and b in the process above. The next couple of examples illustrate this.

Examples

Expand each of the following binomials using Pascal's Triangle.

1. $(2x + 3y)^3$

First, we will need a substitution. Let $a = 2x$ and $b = 3y$.

Then write down the expansion for $(a + b)^3$, replacing a and b when you are done:

To write down $(a + b)^3$, the powers of a go from 3 to 0 as follows:

$$a^3 + a^2 + a^1 + a^0$$

Now, fill in the powers of b , which go from 0 to 3:

$$a^3b^0 + a^2b^1 + a^1b^2 + a^0b^3$$

Now, go to the 4th row in Pascal's Triangle to get the coefficients:

$$1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

Simplify:

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, replace a with $2x$ and b with $3y$:

$$(2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$$

Simplify once again:

$$8x^3 + 36x^2y + 54xy^2 + 27y^3$$

2. $(x - 4y)^5$

First, we will need a substitution. Let $a = x$ and $b = -4y$.

Then write down the expansion for $(a + b)^5$, replacing a and b when you are done:

To write down $(a + b)^5$, the powers of a go from 5 to 0 as follows:

$$a^5 + a^4 + a^3 + a^2 + a^1 + a^0$$

Now, fill in the powers of b , which go from 0 to 5:

$$a^5b^0 + a^4b^1 + a^3b^2 + a^2b^3 + a^1b^4 + a^0b^5$$

Now, go to the 6th row in Pascal's Triangle to get the coefficients:

$$1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

Simplify:

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Now, replace a with x and b with $-4y$:

$$x^5 + 5x^4(-4y) + 10x^3(-4y)^2 + 10x^2(-4y)^3 + 5x(-4y)^4 + (-4y)^5$$

Simplify once again:

$$x^5 - 20x^4y + 160x^3y^2 - 640x^2y^3 + 1280xy^4 - 1024y^5$$

So, that is how you can use Pascal's Triangle to expand a binomial! If the powers are very large, you may not want to write down your polynomial using Pascal's Triangle because it takes a long time to get to the 100th or 1000th row. There is another way that we can write our polynomials using summation notation, which takes up far less room. We can expand the polynomial from the summation formula if we choose to. This formula is called the Binomial Theorem. Before we can introduce it, we first need to explore some notation and the concept of factorials, as we will use these in the formula.

Factorials

A factorial is a product of a number with all of the natural numbers smaller than it:

$$n! = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$$

This is read “ n factorial”.

Note: You don't really need the 1 at the end since multiplying by 1 doesn't change anything.

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Examples

Simplify each of the following factorial expressions.

$$1. \quad 3! = 3 \cdot 2 \cdot 1 = 6$$

$$2. \quad \frac{7!}{5!} = \frac{7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 42$$

Note that we do not really need to expand the factorial all the way out in the numerator if we have another factorial in the

denominator because we can cancel all of the factors in the smaller factorial. For that reason, you could just expand the larger factorial down to the smaller factorial and cancel the factorials:

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

This is very helpful when the numbers are large, as in the next example.

$$3. \quad \frac{120!}{118!} = \frac{120 \cdot 119 \cdot \cancel{118!}}{\cancel{118!}} = 14,280$$

We can have many factorials in one problem.

$$4. \quad \frac{8!3!}{6!} = \frac{8 \cdot 7 \cdot \cancel{6!} \cdot 3 \cdot 2 \cdot 1}{\cancel{6!}} = 336$$

$$5. \quad \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot \overset{3}{\cancel{8 \cdot 7 \cdot 6!}}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1 \cdot 6!}} = 210$$

The form of example 5 occurs often in probability theory and it also occurs in the binomial formula, so we will discuss it a bit more. The form $\frac{n!}{k!(n-k)!}$ is often denoted by $\binom{n}{k}$, which is read “n choose k”. In probability theory, this formula will give you the number of possible combinations of n things taken k at a time, or in other words, the number of ways to choose k things from a set of n things. You could use this formula to calculate your odds of winning a lottery, for example. If there are 60 numbers to choose from and you are choosing 5 of those

numbers, the number of possibilities would be “60 choose 5” or $\binom{60}{5}$, which you can calculate to be:

$$\frac{60!}{5!(60-5)!} = \frac{60!}{5!55!} = \frac{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56 \cdot 55!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 55!} = 5,461,512$$

Since there are 5,461,512 possible combinations of 5 numbers, your chances of getting the winning combination are 1 in 5,461,512.

So when you see the notation $\binom{n}{k}$, you know what it means now. We will see this notation as part of the coefficient in the Binomial Theorem.

Two conventions to note about factorials: $1! = 1$ and $0! = 1$. The first one is probably clear, but the second one is strange, just like the exponent rule $b^0 = 1$. If you remember, the exponent rule came from a pattern we obtained by allowing the power on b to keep getting smaller. We noticed we kept dividing by b each time, until we obtained 1. The logic is very similar here, but it goes beyond the scope of this course and requires a deeper understanding of set theory, so we omit it here hoping that the reader is willing to accept it as a part of the definition of factorial.

The Binomial Theorem

For any positive integer n ,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Isn't this a very concise way to write down a binomial expansion?

If we expanded it out, we could write it as follows:

$$(a + b)^n = \binom{n}{0} a^{n-0} b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^{n-(n-1)} b^{n-1} + \binom{n}{n} a^{n-n} b^n$$

Which simplifies to a nicer form:

$$= a^n + \binom{n}{1} a b^{n-1} + \binom{n}{2} a^2 b^{n-2} + \cdots + \binom{n}{n-1} a b + a^n$$

But, all we really need to know how to do is to compute the coefficients and allow the powers of a to decrease and the powers of b to increase, just as we did before. This is just a different way to get those values we got from Pascal's Triangle!

Examples

Expand each of the following binomials using the Binomial Theorem.

$$\begin{aligned} 1. \quad (a + b)^6 &= \sum_{k=0}^6 \binom{6}{k} a^{6-k} b^k \\ &= \binom{6}{0} a^{6-0} b^0 + \binom{6}{1} a^{6-1} b^1 + \binom{6}{2} a^{6-2} b^2 + \binom{6}{3} a^{6-3} b^3 + \binom{6}{4} a^{6-4} b^4 + \binom{6}{5} a^{6-5} b^5 + \binom{6}{6} a^{6-6} b^6 \\ &= \binom{6}{0} a^6 + \binom{6}{1} a^5 b + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3 + \binom{6}{4} a^2 b^4 + \binom{6}{5} a b^5 + \binom{6}{6} b^6 \end{aligned}$$

Computing each coefficient:

$$\binom{6}{0} = \frac{6!}{0!(6-0)!} = \frac{6!}{0!6!} = 1$$

$$\binom{6}{1} = \frac{6!}{1!(6-1)!} = \frac{6!}{1!5!} = \frac{6 \cdot 5!}{5!} = 6$$

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot 4!}{2 \cdot 4!} = 15$$

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3 \cdot 2 \cdot 3!} = 20$$

$$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2} = 15$$

$$\binom{6}{4} = \frac{6!}{5!(6-5)!} = \frac{6!}{5!1!} = \frac{6 \cdot 5!}{5!} = 6$$

$$\binom{6}{6} = \frac{6!}{6!(6-6)!} = \frac{6!}{6!0!} = 1$$

Replacing the coefficients gives us:

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$2. (2x + y)^4 = \sum_{k=0}^4 \binom{4}{k} (2x)^{4-k} y^k$$

$$= \binom{4}{0} (2x)^{4-0} y^0 + \binom{4}{1} (2x)^{4-1} y^1 + \binom{4}{2} (2x)^{4-2} y^2 + \binom{4}{3} (2x)^{4-3} y^3 + \binom{4}{4} (2x)^{4-4} y^4$$

$$= \binom{4}{0} (2x)^4 + \binom{4}{1} (2x)^3 y + \binom{4}{2} (2x)^2 y^2 + \binom{4}{3} 2x y^3 + \binom{4}{4} y^4$$

Computing each coefficient:

$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{4!}{0!4!} = 1$$

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{1!3!} = \frac{4 \cdot 3!}{3!} = 4$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2!}{2 \cdot 2!} = 6$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{4 \cdot 3!}{3!} = 4$$

$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{4!}{4!0!} = 1$$

Replacing the coefficients gives us:

$$= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4 \cdot 2xy^3 + y^4$$

And finally, simplify:

$$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$$