7.4 Arithmetic and Geometric Series

When we add up the terms of arithmetic and geometric sequences, we call them *arithmetic series* and *geometric series*, respectively. We have formulas that we can use when evaluating these types of series. This is very helpful, especially if there are many terms to add! We can recognize an arithmetic series by the linear sequence it is summing (i.e. the general term will be linear in the index). We can also recognize a geometric series by the exponential sequence it is summing (i.e. the general term will be exponential in the index).

<u>Arithmetic Series</u>

One formula that will work for an arithmetic series (there are actually two formulas) was derived by Carl Friedrich Gauss (1777-1855):

$$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$$

So if you add the first and the last terms together and then multiply by half of the number of terms, you will be done! How is this possible?

Gauss noticed that when adding the numbers from 1 to 100, a shortcut emerged if he paired up the numbers in a certain way – pair the first with the last, the second with the second to last, etc. – the sum is the same for these pairs and you will have 50 pairs (since you paired up 100 things, you divide by 2), so multiply by 50:

$$1 + 2 + 3 + 4 + 5 + \cdots + 96 + 97 + 98 + 99 + 100$$

$$3 + 98 = 101$$

$$2 + 99 = 101$$

$$1 + 100 = 101$$

So Gauss arrived at the fact that the first 100 numbers add to be

$$\frac{100}{2}(1 + 100) = 5050$$

and if we generalize this to any sequence, we get the formula above. We could prove this formula using the same method:

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-3} + a_{n-2} + a_{n-1} + a_n$$

$$a_3 + a_{n-2}$$

$$a_2 + a_{n-1}$$

$$a_1 + a_n$$

We just need to show that we get the same sum when we pair the terms this way. Then they should all equal $a_1 + a_n$. We can show this is true if we replace each term using the formula $a_k = a_1 + (k-1)d$ and do a little simplifying:

$$\begin{aligned} a_2 + a_{n-1} &= (a_1 + d) + (a_1 + (n-2)d)) \\ &= 2a_1 + d + nd - 2d \\ &= 2a_1 + nd - d \\ &= 2a_1 + (n-1)d \\ &= a_1 + a_1 + (n-1)d \\ &= a_1 + a_n \end{aligned}$$

The same logic will apply for each of the middle sums. Therefore,

$$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$$

Now, let's use this formula to compute some sums.

<u>Examples</u>

1. Evaluate the following arithmetic series:

$$\sum_{k=1}^{30}(3k+1)$$

Here, we need to figure out what to plug into the formula for $n, a_1,$ and a_n. Looking at the form, we should be able to identify that $n = 30$.

We can figure out a_1 and a_{30} by plugging the values $k = 1$ and $k = 30$ into the general term $a_k = 3k + 1$.

$$a_1 = 3 \cdot 1 + 1 = 4$$
$$a_{30} = 3 \cdot 30 + 1 = 91$$

$$\sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$$

$$\Rightarrow \quad \sum_{k=1}^{30}(3k+1) = \frac{30}{2}(4+91)$$

$$= 15(95)$$

$$= 1{,}425$$

2. Evaluate the following arithmetic series:

$$\sum_{i=1}^{13}(7i-4)$$

We need to figure out what to plug into the formula for $n, a_1,$ and a_n. Looking at the form, we should be able to identify that $n = 13$.

We can figure out a_1 and a_{13} by plugging the values $i = 1$ and $i = 13$ into the general term $a_i = 7i - 4$.

$$a_1 = 7 \cdot 1 - 4 = 3$$
$$a_{13} = 7 \cdot 13 - 4 = 87$$

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

$$\Rightarrow \quad \sum_{i=1}^{13}(7i - 4) = \frac{13}{2}(3 + 87)$$

$$= \frac{13}{2}(90)$$

$$= 13(45)$$

$$= 585$$

Sometimes the questions can be worded differently, as in the next example.

3. Find S_{100} for the sequence $\{a_i\}$ where $a_i = 5i + 7$

Even if we were not told that this was an arithmetic sequence, we could tell by looking at its form since it is linear. Then we know we can use the formula.

We need to figure out what to plug into the formula for $n, a_1,$ and a_n. Looking at the form, we should be able to identify that $n = 100$.

We can figure out a_1 and a_{100} by plugging the values $i = 1$ and $i = 100$ into the general term $a_i = 5i + 7$.

$$a_1 = 5 \cdot 1 + 7 = 12$$
$$a_{100} = 5 \cdot 100 + 7 = 507$$

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

$$\Rightarrow \qquad \sum_{i=1}^{100}(5i + 7) = \frac{100}{2}(12 + 507)$$

$$= 50(519)$$

$$= 25{,}950$$

4. Evaluate the following series:

$$\sum_{j=5}^{32}(10j - 4)$$

For this example, we can see that the general term is linear, so this is an arithmetic series, but the series is not starting at 1, but at 5 instead. We need to be careful here because the formula only applies when we are adding the first n terms, which means we have to be starting at 1 to use it. Since we really do not wish to write out all of the terms from a_5 to a_{32} and add them up, we should look for a way to write this series as a combination of sums that begin at 1 so that we can use the formula on those pieces. It is really not difficult if you think of adding ALL of the terms from a_1 to a_{32} and then subtracting off the first four terms a_1 to a_4. That should leave us with the terms we want to add – a_5 to a_{32}:

$$\sum_{j=5}^{32}(10j-4) = \sum_{j=1}^{32}(10j-4) - \sum_{j=1}^{4}(10j-4)$$

Now, we can use the formula on each piece since they both start at 1 and then subtract:

For the first piece: $\sum_{j=1}^{32}(10j-4)$

$$a_1 = 10 \cdot 1 - 4 = 6$$
$$a_{32} = 10 \cdot 32 - 4 = 316$$

⇨ $$\sum_{j=1}^{32}(10j-4) = \frac{32}{2}(6+316)$$

$$= 16(322)$$

$$= 5{,}152$$

For the second piece: $\sum_{j=1}^{4}(10j-4)$

$$a_1 = 10 \cdot 1 - 4 = 6$$
$$a_4 = 10 \cdot 4 - 4 = 36$$

⇨ $$\sum_{j=1}^{4}(10j-4) = \frac{4}{2}(6+36)$$

$$= 2(42)$$

$$= 84$$

Therefore, $\sum_{j=5}^{32}(10j - 4) = 5152 - 84 = 5068$

Geometric Series

There is also a formula for adding the first n terms in a geometric series. This formula is as follows:

$$S_n = \sum_{k=1}^{n} a_1 r^{k-1} = \frac{a_1(1 - r^n)}{1 - r}$$

We can prove this formula using a trick (by subtracting r times the sum to both sides, we can get all but two of the terms to cancel out on one side):

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} + a_n$$

⇨ $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$

⇨ $rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^{n-1} + a_1 r^n$

No subtracting rS_n from S_n, we see that all of the terms drop out except for the first and the last term:

⇨ $S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$

$-rS_n = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - \cdots - a_1 r^{n-1} - a_1 r^n$

⇨ $S_n - rS_n = a_1 - a_1 r^n$

Now, solving for S_n:

⇨ $S_n(1-r) = a_1 - a_1 r^n$

⇨ $\frac{S_n(1-r)}{1-r} = \frac{a_1 - a_1 r^n}{1-r}$

⇨ $S_n = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1(1-r^n)}{1-r}$

Now, let's use this formula in some examples.

Examples

5. Evaluate the following geometric series:

$$\sum_{k=1}^{10} 3\left(\frac{1}{2}\right)^{k-1}$$

Here, we can see that $n = 10$.

We can identify $a_1$and r in the general term $a_1 r^{k-1}$ by inspection ($a_1 = 3\ and\ r = \frac{1}{2}$ in the form for a_k in the sum above) or we can plug the value $k = 1$ into the general term $a_k = 3\left(\frac{1}{2}\right)^{k-1}$.

$$a_1 = 3\left(\frac{1}{2}\right)^{1-1} = 3\left(\frac{1}{2}\right)^{0} = 3 \cdot 1 = 3$$

Now the formula says:

$$\sum_{k=1}^{n} a_k = \frac{a_1(1-r^n)}{1-r}$$

$$\Rightarrow \qquad \sum_{k=1}^{10} 3\left(\frac{1}{2}\right)^{k-1} = \frac{3\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$= \frac{3\left(1-\frac{1}{1024}\right)}{1-\frac{1}{2}}$$

$$= \frac{3\left(\frac{1023}{1024}\right)}{\frac{1}{2}}$$

$$= 3 \cdot \frac{1023}{1024} \cdot 2$$

$$= \frac{3069}{512}$$

6. Evaluate the following geometric series:

$$\sum_{j=1}^{9} 4(3)^{j-1}$$

Here, we can see that $n = 9$.

We can identify $a_1$and r in the general term $a_1 r^{j-1}$ by inspection ($a_1 = 4 \ and \ r = 3$ in the form for a_j in the sum above) or we can plug the value $j = 1$ into the general term $a_j = 4(3)^{j-1}$.

$$a_1 = 4(3)^{1-1} = 4(3)^0 = 4 \cdot 1 = 4$$

Now the formula says:

$$\sum_{j=1}^{n} a_j = \frac{a_1(1-r^n)}{1-r}$$

⇨ $$\sum_{j=1}^{9} 4(3)^{j-1} = \frac{4(1-3^9)}{1-3}$$

$$= \frac{4(1-19{,}683)}{-2}$$

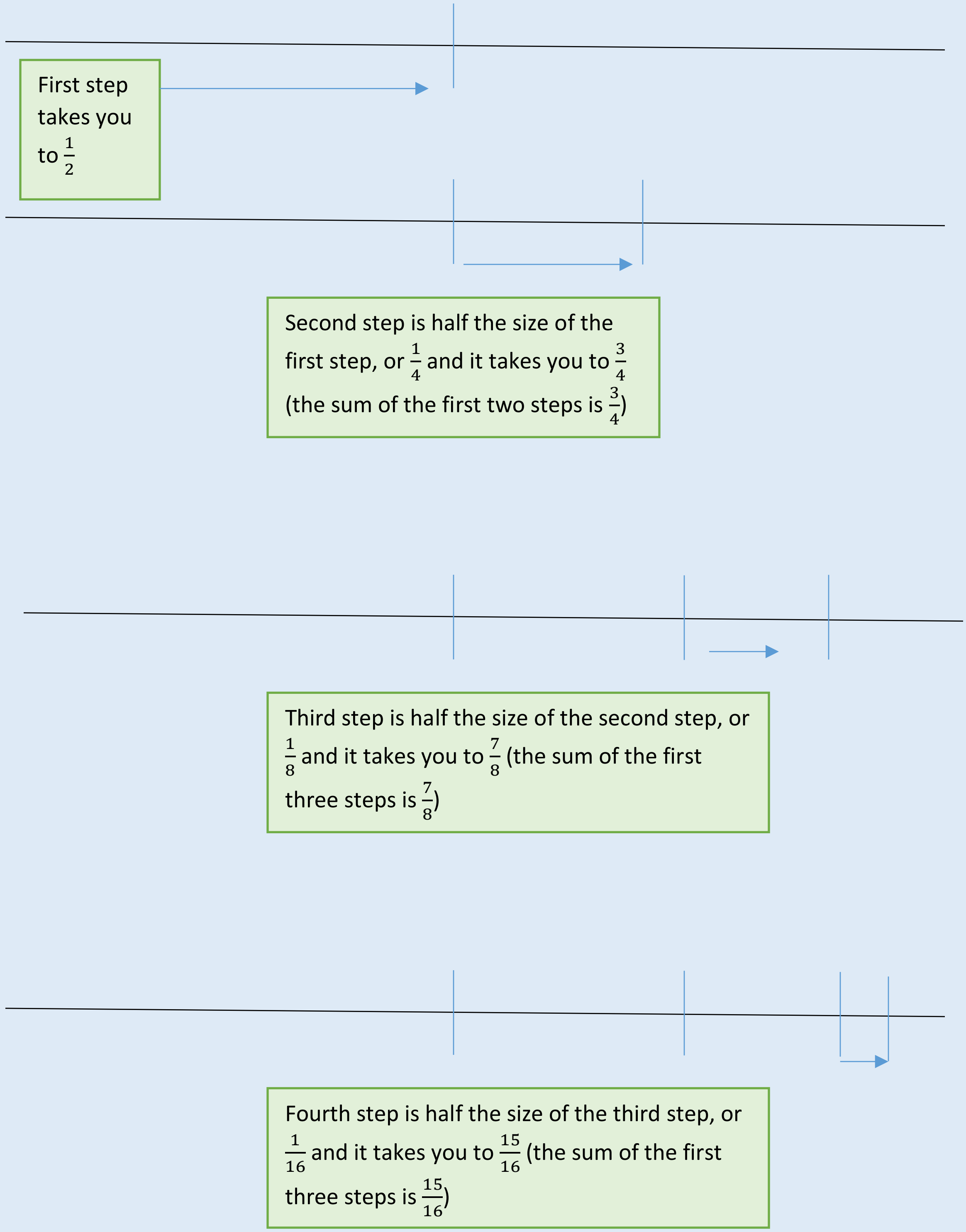
$$= -2(-19{,}682)$$

$$= 39{,}364$$

Notice that we only thought about finite sums for arithmetic series. This is because if you add up infinitely many arithmetic terms, you will either get ∞ or $-\infty$. We will, however, consider infinite sums for geometric series. Sometimes they add up to a finite number and sometimes they do not. When a series does sum to a finite value, we say that it *converges* to that value. If it does not add up to a finite value, we say that the series *diverges*. Arithmetic terms are too "large" to allow convergence of the series when we add them. We can tell whether or not a geometric series will converge by looking at the common ratio r. If the ratio is "small enough", then the terms we are adding will get small enough fast enough for the series to converge. We will make this more precise after an intuitive and purely hypothetical example.

Suppose you are going to walk across a room of length 1 (for ease of computation) in the following way: Each step that you take has to be half of the distance of your last step. The first step you take will move you halfway across the room. Will you ever get to the other side of the room?

A picture might help:



Can you see that you will never actually arrive at the other side? Another way to formulate the same problem is that you keep moving half the distance that is left. Can you tell that when you add up the distances traveled, you are approaching the distance of 1? We can think of the size of each step as a term in an infinite geometric sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$ and if we add up these steps, we will get the distance traveled, which approaches 1 as we take more steps. (This is the idea of a *limit*, which you will explore deeply in a calculus course.) We can write the terms of our geometric sequence in the form $a_1 r^{k-1}$ as $\frac{1}{2}\left(\frac{1}{2}\right)^{k-1}$. Now adding up the terms (taking infinitely many steps:

$$\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} = 1$$

So now we have an example of an infinite series that converges to a finite number. We also have a formula for summing infinite geometric series when they converge:

$$S_\infty = \frac{a_1}{1-r} \text{ when } |r| < 1$$

Always check r before using this formula since if $|r| \geq 1$, the series will diverge. Proving this formula requires an understanding of limits, but the general idea is that if you take the formula $S_n = \frac{a_1(1-r^n)}{1-r}$ and you allow n to get very large, the value r^n will get very small and eventually go away.

7. Evaluate the following infinite geometric series:

$$\sum_{k=1}^{\infty} 4\left(\frac{1}{3}\right)^{k-1}$$

Since this is an infinite geometric series, look at the value of r first. Here $r = \frac{1}{3}$ and since $\left|\frac{1}{3}\right| < 1$, this series converges and we can use the formula.

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{4}{1-\frac{1}{3}} \\ &= \frac{4}{\frac{2}{3}} \\ &= 4 \cdot \frac{3}{2} \\ &= 6 \end{aligned}$$

8. Evaluate the following infinite geometric series:

$$\sum_{k=1}^{\infty} 4\left(-\frac{5}{2}\right)^{k-1}$$

Since this is an infinite geometric series, look at the value of r first. Here $r = -\frac{5}{2}$ and since $\left|-\frac{5}{2}\right| = \frac{5}{2} > 1$, this series diverges.