7.3 Introduction to Series

A *series* is really just a sum of items, typically the terms of a sequence.

While a sequence is a list, a series is the sum of that list:

Sequence: $\{0,1,1,2,3,5,8,13 \ldots.\}$

Series: $0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + \cdots.$

We use a special symbol to denote the sum. It is a Greek letter called *sigma* and it looks like this: $\sum$

This symbol means to add, but we also have to specify what we are adding, so there is a bit more notation to understand before we can begin a more in depth discussion of series. If we let a_k denote the general term of a sequence, then we can write down the sum of the first n terms of this sequence in the following way:

$$\sum_{k=1}^{n} a_k$$

This is read "The sum of a_k as k goes from *1* to *n*" and it is called a *partial sum* of the sequence. In general, the bottom number where to start adding (which term to begin with) and the top tells us where to stop (which term to end with).

For example,

$$\sum_{k=3}^{7} a_k = a_3 + a_4 + a_5 + a_6 + a_7$$

This means you would add together the third through the seventh terms.

Consider the Fibonacci Sequence: $\{a_k\} = \{0,1,1,2,3,5,8,13 \ldots.\}$. If we wanted to add up the first ten terms, we would get the following:

$$\sum_{k=1}^{10} a_k = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 = 88$$

If we wanted to add up 100 terms, this would take up quite a bit of space and time. It would be nice if there was a formula we could use to add up the terms of a series, and it turns out that for arithmetic series and for geometric series, there are formulas we can use. For some other types of series, there are formulas you will learn in more advanced math courses. In this section, we will focus on understanding the meaning of summation notation and we will do the calculations rather than using a formula.

Examples

1. Evaluate the series:

$$\sum_{k=1}^{7} k^2$$

Here, we will add up the terms of the sequence $a_k = k^2$ as k goes from 1 to 7. We could calculate the terms of the sequence first and then add them all up:

$$a_1 = 1^2 = 1$$

$$a_2 = 2^2 = 4$$

$$a_3 = 3^2 = 9$$

$$a_4 = 4^2 = 16$$

$$a_5 = 5^2 = 25$$

$$a_6 = 6^2 = 36$$

$$a_7 = 7^2 = 49$$

$$\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

$$= 1 + 4 + 9 + 16 + 25 + 36 + 49$$

$$= 140$$

So the sum is 140.

2. Evaluate the series:

$$\sum_{k=3}^{11} (2k - 1)$$

Here, we will add up the terms of the sequence $a_k = 2k - 1$ as k goes from 3 to 11.

$$a_3 = 2(3) - 1 = 5$$

$$a_4 = 2(4) - 1 = 7$$

$$a_5 = 2(5) - 1 = 9$$

$$a_6 = 2(6) - 1 = 11$$

$$a_7 = 2(7) - 1 = 13$$

$$a_8 = 2(8) - 1 = 15$$

$$a_9 = 2(9) - 1 = 17$$

$$a_{10} = 2(10) - 1 = 19$$

$$a_{11} = 2(11) - 1 = 21$$

$$\sum_{k=3}^{11}(2k-1) = 5+7+9+11+13+15+17+19+21$$

$$= 117$$

So the sum is 117.

You may recognize the sequence as an arithmetic sequence that we are adding. This is true. You will have a formula in the next section for series such as these.

3. Evaluate the series:

$$\sum_{n=2}^{6} n!$$

Here, we will add up the terms of the sequence $a_n = n!$ as n goes from 3 to 11.

$$a_2 = 2! = 2 \cdot 1 = 2$$

$$a_3 = 3! = 3 \cdot 2 \cdot 1 = 6$$

$$a_4 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$a_5 = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$a_6 = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$\sum_{n=2}^{6} n! = 2+6+24+120+720$$

$$= 872$$

So the sum is 872.

Series can also result in expressions such as polynomials. If you have a variable in your expression other than the index, you will have an expression as your sum. The next example illustrates this.

4. Expand the series:

$$\sum_{n=1}^{5}(n+1)!\,x^n$$

Here, we will write terms of the sequence $a_n = (n+1)!\,x^n$ as n goes from 1 to 5. Keep in mind that we are plugging in for n, not x.

$$a_1 = (1+1)!\,x^1 = 2!\,x = 2x$$

$$a_2 = (2+1)!\,x^2 = 3!\,x^2 = 6x^2$$

$$a_3 = (3+1)!\,x^3 = 4!\,x^3 = 24x^3$$

$$a_4 = (4+1)!\,x^4 = 5!\,x^4 = 120x^4$$

$$a_5 = (5+1)!\,x^5 = 6!\,x^5 = 720x^5$$

$$\sum_{n=1}^{5}(n+1)!\,x^n = 2x + 6x^2 + 24x^3 + 120x^4 + 720x^5$$