7.2 Arithmetic and Geometric Sequences

Sequences that have the addition pattern between the terms are called *arithmetic sequences* and sequences that have the multiplication pattern between the terms are called *geometric sequences*. Below is an example of each type:

Arithmetic: $\{5,8,11,14,17,20,\ldots\}$

$+3 \quad +3 \quad +3$

The number that is repeatedly added is called the "common difference" and is usually denoted by the letter d.

We can see that in order to get from one term to the next term, we are adding 3 each time. Adding the same thing to get from term to term is the hallmark of an arithmetic sequence.

Geometric: $\{5,15,45,135,,405,\ldots\}$

$\times 3 \quad \times 3 \quad \times 3$

The number that is repeatedly multiplied is called the "common ratio" and is usually denoted by the letter r.

We can see that in order to get from one term to the next term, we are multiplying by 3 each time. Multiplying by the same thing to get from term to term is the hallmark of a geometric sequence.

Now, let's get some practice categorizing the different types of sequences….

<u>Examples</u>

For each of the following sequences, determine whether it is arithmetic, geometric, or neither.

1. $\{3,7,11,15,19,\ldots.\}$

   This sequence is arithmetic, since we are adding the same thing to get from term to term. The common difference is $d = 4$.

2. $\{7,15,23,31,39, \ldots.\}$

This sequence is arithmetic, since we are adding the same thing to get from term to term. The common difference is $d = 8$.

3. $\{3,6,12,24,48, \ldots.\}$

This sequence is geometric, since we are multiplying by the same thing to get from term to term. The common ratio is $r = 2$.

4. $\left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots.\right\}$

This sequence is geometric, since we are multiplying by the same thing to get from term to term. The common ratio is $r = -\frac{1}{2}$.

5. $\{5,10,15,20,25, \ldots.\}$

This sequence is arithmetic, since we are adding the same thing to get from term to term. The common difference is $d = 5$. (Often students think this one is geometric at first because they can think of it as multiplying by something to get to the next term, but note that you are multiplying by something different each time, so it is not geometric.)

6. $\{1,7,14,22,31, \ldots.\}$

This sequence is not arithmetic or geometric, since we are neither adding the same thing each time nor multiplying by the same thing each time.

We know how to find the next term in either an arithmetic or a geometric sequence. What do we do if we want to find the 100th term or the 1000th term of a sequence? We would not want to write down 1000 terms, right? For both arithmetic and geometric sequences, we can come up with a formula that will predict the value of any term. We will begin by considering arithmetic sequences.

It is easiest to discover the formula for the *general term* a_n by beginning with a specific example and generalizing from it.

Consider the arithmetic sequence $\{3,7,11,15,19, \ldots\}$. Let's list the terms of this sequence and look for a pattern:

Term	Pattern
$a_1 = 3$	$= 3 + 0 \cdot 4$
$a_2 = 7 \;\; = 3 + 4$	$= 3 + 1 \cdot 4$
$a_3 = 11 = 3 + 4 + 4$	$= 3 + 2 \cdot 4$
$a_4 = 15 = 3 + 4 + 4 + 4$	$= 3 + 3 \cdot 4$
$a_5 = 19 = 3 + 4 + 4 + 4 + 4$	$= 3 + 4 \cdot 4$
.	
.	
.	
	$a_n = 3 + (n - 1) \cdot 4$

Note that for this sequence, $a_1 = 3$ and $d = 4$.

So, if we wanted to find the 100th term, we could calculate it using this formula.

$$
\begin{aligned}
a_{100} &= 3 + (100 - 1) \cdot 4 \\
&= 3 + 99 \cdot 4 \\
&= 3 + 396 = 399
\end{aligned}
$$

or if you distribute instead:
$= 3 + 400 - 4 = 399$

For any arithmetic sequence, the *general term* a_n can be found using the formula:

$$a_n = a_1 + (n-1)d$$

Examples

Find the 30th term for each of the following arithmetic sequences and then find a formula for the general term.

1. $\{11,8,5,2,-1,\ldots\}$

To calculate the 30th term, we can begin with the formula and plug in values for $a_1, d,$ and n.

$$a_n = a_1 + (n-1)d$$

Note that $a_1 = 11, d = -3,$ and $n = 30$.

$$\begin{aligned} a_{30} &= 11 + (30-1)(-3) \\ &= 11 - 90 + 3 \\ &= -76 \end{aligned}$$

To find the general term, we can begin with the formula and plug in values for a_1 and d.

$$a_n = a_1 + (n-1)d$$

$$a_n = 11 + (n-1)(-3)$$

Now simplify:

$$a_n = 11 - 3n + 3$$

$$a_n = 14 - 3n \text{ or } a_n = -3n + 14$$

Notice that the formula for the general term of an arithmetic sequence always results in a linear expression in the variable n. This is not a coincidence, as you would see if you graph the terms of the sequence as the y values of a function (on the natural numbers), they lie in a straight line. That is one of the ways we can recognize an arithmetic sequence if we are given the formula rather than a list of terms. This will help us in a future section when we are trying to determine whether a *series* is arithmetic or geometric.

2. $\{4,9,14,19,24,\ldots\}$

To calculate the 30th term, we can begin with the formula and plug in values for $a_1, d,$ and n.

$$a_n = a_1 + (n-1)d$$

Note that $a_1 = 4, d = 5,$ and $n = 30$.

$$\begin{aligned} a_{30} &= 4 + (30-1)(5) \\ &= 4 + 150 - 5 \\ &= 149 \end{aligned}$$

To find the general term, we can begin with the formula and plug in values for a_1 and d.

$$a_n = a_1 + (n-1)d$$

$$a_n = 4 + (n-1)5$$

Now simplify:

$$a_n = 4 + 5n - 5$$

$$a_n = -1 + 5n \text{ or } a_n = 5n - 1$$

3. $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots\right\}$

To calculate the 30th term, we can begin with the formula and plug in values for $a_1, d,$ and n.

$$a_n = a_1 + (n-1)d$$

Note that $a_1 = 0, d = \frac{1}{4}$, and $n = 30$.

$$a_{30} = 0 + (30-1)\frac{1}{4}$$
$$= \frac{29}{4}$$

To find the general term, we can begin with the formula and plug in values for a_1 and d.

$$a_n = a_1 + (n-1)d$$

$$a_n = 0 + (n-1)\frac{1}{4}$$

Now simplify:

$$a_n = \frac{1}{4}n - \frac{1}{4}$$

Once again, not that the general term is linear in n.

Geometric sequences also have a pattern that can be predicted for the general term. We will come up with the formula for this type of sequence as well. You will see that the pattern does not give us a linear form as did the arithmetic sequence, but rather an exponential form (where the variable is in the exponent).

Consider the geometric sequence $\{4,12,36,108,324, \ldots\}$. Let's list the terms of this sequence and look for a pattern:

Term	Pattern
$a_1 = 4$	$= 4 \cdot 3^0$
$a_2 = 12 \quad = 4 \cdot 3$	$= 4 \cdot 3^1$
$a_3 = 36 \quad = 4 \cdot 3 \cdot 3$	$= 4 \cdot 3^2$
$a_4 = 108 = 4 \cdot 3 \cdot 3 \cdot 3$	$= 4 \cdot 3^3$
$a_5 = 324 = 4 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	$= 4 \cdot 3^4$
.	
.	
.	
	$a_n = 4 \cdot 3^{n-1}$

Note that for this sequence, $a_1 = 4$ and $d = 3$.

So, if we wanted to find the 10th term, we could calculate it using this formula.

$$\begin{aligned} a_{10} &= 4 \cdot 3^{10-1} \\ &= 4 \cdot 3^9 \\ &= 4 \cdot 19{,}683 = 78{,}732 \end{aligned}$$

For any geometric sequence, the *general term* a_n can be found using the formula:

$$a_n = a_1 r^{n-1}$$

Examples

Find the 10th term for each of the following arithmetic sequences and then find a formula for the general term.

4. $\{3,6,12,24,48,\ldots\}$

To calculate the 10th term, we can begin with the formula and plug in values for $a_1, r,$ and n.

$$a_n = a_1 r^{n-1}$$

Note that $a_1 = 3, r = 2,$ and $n = 10$.

$$\begin{aligned} a_{10} &= 3 \cdot 2^{10-1} \\ &= 3 \cdot 2^9 \\ &= 3 \cdot 512 \\ &= 1536 \end{aligned}$$

To find the general term, we can begin with the formula and plug in values for a_1 and r.

$$a_n = a_1 r^{n-1}$$

$$a_n = 3 \cdot 2^{n-1}$$

Note that exponents come before multiplication in the order of operations, so we cannot simplify this any more than it is. Also, notice that the formula for the general term of an geometric sequence always results in an exponential expression in the variable n. If you were to graph the terms of the sequence as the y values of a function (on the natural numbers), they lie on an exponential curve. That is one of the ways we can recognize an geometric sequence if we are given the formula rather than a list of terms. Again, this will help us in the next lesson.

5. $\left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots\right\}$

To calculate the 10th term, we can begin with the formula and plug in values for $a_1, r,$ and n.

$$a_n = a_1 r^{n-1}$$

Note that $a_1 = 1, r = \frac{2}{3},$ and $n = 10$.

$$\begin{aligned} a_{10} &= 1 \cdot \left(\frac{2}{3}\right)^{10-1} \\ &= \left(\frac{2}{3}\right)^{9} \\ &= \frac{2^9}{3^9} \\ &= \frac{512}{19{,}683} \end{aligned}$$

To find the general term, we can begin with the formula and plug in values for a_1 and r.

$$a_n = a_1 r^{n-1}$$

$$a_n = 1 \cdot \left(\frac{2}{3}\right)^{n-1}$$

Or simplified:

$$a_n = \left(\frac{2}{3}\right)^{n-1}$$

6. Find the twelfth term of the arithmetic sequence with a first term of 3 and a common difference of -6.

To calculate the 12th term, we can begin with the formula for the general term of an arithmetic sequence and plug in values for $a_1, d,$ and n.

$$a_n = a_1 + (n-1)d$$

Note that $a_1 = 3, d = -6,$ and $n = 12$.

$$\begin{aligned} a_{12} &= 3 + (12-1)(-6) \\ &= 3 + 11(-6) \\ &= 3 - 66 \\ &= -63 \end{aligned}$$

7. Find the eighth term of the geometric sequence with a first term of 10 and a common ration of $-\frac{1}{2}$.

To calculate the 8th term, we can begin with the formula for the general term of a geometric sequence and plug in values for $a_1, r,$ and n.

$$a_n = a_1 r^{n-1}$$

Note that $a_1 = 10, r = -\frac{1}{2},$ and $n = 8$.

$$\begin{aligned} a_8 &= 10 \cdot \left(-\frac{1}{2}\right)^{8-1} \\ &= 10\left(-\frac{1}{2}\right)^7 \end{aligned}$$

$$= 10 \cdot -\frac{1^7}{2^7}$$
$$= -\frac{10}{128}$$
$$= -\frac{5}{64}$$

Sequences can also be used to model some natural processes as we will see in the next example.

8. The initial size of a virus culture is 15 units and it quadruples every day. Find the general term of the sequence that models the culture's size.

First, determine whether the sequence is arithmetic or geometric. You can write down a few terms if that helps to see that it is a geometric sequence (we are multiplying by 4 each time):

$$\{15, 60, 240, \ldots\}$$

To find the general term, we can begin with the formula and plug in values for a_1 and r.

$$a_n = a_1 r^{n-1}$$

$$a_n = 15 \cdot 4^{n-1}$$