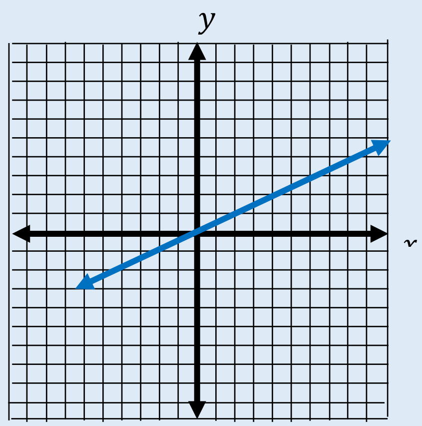
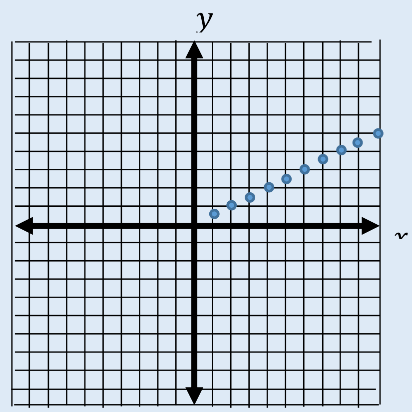
7.1 Introduction to Sequences

A *sequence* is really just a list of items, or *terms*, in a particular order. Your grocery list could be considered a sequence, for example. In this lesson, we are looking at sequences of numbers, and in particular sequences of numbers with patterns between them.

Sequences can also be thought of in relation to functions, and they often are in the field of statistics. In particular, a sequence can be thought of as the range of any function when the domain of that function is restricted to the natural (or counting) numbers. Consider the function $y = \frac{1}{2}x$, for example.





If we restrict the domain of the function $f(x) = \frac{1}{2}x$ to the natural numbers $\{1,2,3,4,5, \dots\}$, then we get a subset of the line consisting of points. The y values of those points are the sequence that results: $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \dots.\right\}$. Notice that each term of the sequence can be thought of as a member of the range of this function by simply plugging the natural numbers into the function: $f(1) = \frac{1}{2}, f(2) = 1. f(3) = \frac{3}{2}, etc.$

Any sequence can be thought of in this way and this motivates the notation that we use for the terms of a sequence $\{a_1, a_2, a_3, a_4, a_5, \dots\}$

Note: In statistics, you will often work backwards and start with a set of points (a relationship between two variables) and try to find a line that best fits it. This is called correlation / regression analysis. Your points will likely not lie in a perfectly straight line, but the closer they are, the more established the relationship is between those variables. Any arithmetic sequence is linear.

Terminology alert: First, notice that we are using the words "term" in a different way than we used it for polynomials. The terms of a sequence will be added together when we talk about series and then the connection will be clearer.

Also, there is quite a bit of notation introduced in the next two lessons that can be a stumbling block for students, so we will intermittently stop to talk about it. For this lesson, you will need to know the following notation:

A sequence can be referred to as $\{a_n\}$ or $\{a_1, a_2, a_3, a_4, a_5, ...\}$ where a_1 just means the first term of the sequence, a_2 means the second term of the sequence,…. a_i means the i^{th}term of the sequence, etc. The subscript is called the *index* of the sequence. It doesn't matter what letter you use for the index. For example, $\{a_n\}$ means the same thing as $\{a_k\}$which means the same thing as $\{a_i\}$, just like we can call variables any letter that we like.

Examples

Try to find the pattern for each of the following sequences. Predict the next 3 terms.

1. $\{5, 7,9,11,13, ...\}$

   If you look at each term carefully, you will see that 2 is added repeatedly to get from one term to the next. This is the pattern. The next three terms will be 15, 17, and 19.

   When the same number is repeatedly added to get to the next term, we call this an **arithmetic sequence**. We will study these more in the next section and you will learn how to find the formula for the nth term of any arithmetic sequence. The general term here can be described by $a_n = 3 + 2n$ if starting at $n = 1$. (Note: If you start at $n = 0$, you could use the formula $a_n = 5 + 2n$, but it is more common to start counting at 1 than at 0). Plug in a few values for n to see this:

$$a_1 = 3 + 2(1) = 5$$

$$a_2 = 3 + 2(2) = 7$$

$$a_3 = 3 + 2(3) = 9$$

$$a_4 = 3 + 2(4) = 11$$

And so on….

2. $\{2, 6, 18, 54, \ldots\}$

Considering each term and how to get from one term to the next, you will notice that 3 is multiplied repeatedly. This is the pattern. The next three terms are 162, 486, and 1458.

When the same number is repeatedly multiplied to get to the next term, we call this a **geometric sequence**. We will also study these more in the next section and you will learn how to find the formula for the nth term of any geometric sequence as well. The general term here can be described by $a_n = 2 \cdot 3^{n-1}$ if starting at $n = 1$. (If you start at $n = 0$, you could use the formula $a_n = 2 \cdot 3^n$). Plug in a few values for n to see this:

$$a_1 = 2 \cdot 3^{1-1} = 2 \cdot 3^0 = 2 \cdot 1 = 2$$

$$a_2 = 2 \cdot 3^{2-1} = 2 \cdot 3^1 = 2 \cdot 3 = 6$$

$$a_3 = 2 \cdot 3^{3-1} = 2 \cdot 3^2 = 2 \cdot 9 = 18$$

$$a_4 = 2 \cdot 3^{4-1} = 2 \cdot 3^3 = 2 \cdot 27 = 54$$

And so on….

3. $\{1, 4, 9, 16, 25, \ldots\}$

The pattern here is neither adding nor multiplication. If you look closely, you will see that each term is a perfect square. We have $\{1^2, 2^2, 3^2, 4^2, 5^2, \ldots\}$, or you could write this more generally as $a_n = n^2$. The next three terms will be 36, 49, and 64.

4. $\{1, 2, 6, 24, 120, \ldots\}$

This one might be difficult to find the pattern for unless you are familiar with factorials, but you may be able to see that each term is multiplied by the next higher number (e.g. x2, x3, x4, x5 etc.). So the next term would be the previous term multiplied by 6, and so on. The name for this pattern is factorial and is denoted with an exclamation point!

$$n! = n(n-1)(n-2) \cdots\cdots 2 \cdot 1$$

For example,

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Basically, a factorial tells you to multiply the number by the product of all previous whole numbers.

So the next three terms are $6!, 7!,$ and 8! as follows:

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 120 = 720$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7 \cdot 720 = 5040$$

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 5040 = 40{,}320$$

We see from the last example that sequences can grow very quickly, just as functions can. A sequence can represent any type of function such as exponential (geometric), logarithmic, absolute value, linear (arithmetic), factorial, rational, etc. Learning to recognize the pattern going from one term to the next will be important when working with sequences. You will learn methods for finding the general term for specific patterns as you move through your math courses. In this course, we focus on the basic idea of sequences and patterns. We also explore the arithmetic and geometric patterns in the next section.

5. $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots.\}$

   Look carefully at this one. You may have seen this sequence before. It is called the Fibonacci Sequence and we see it in nature describing things like the patterns in the growth cycle of leaves on plants and petals on flowers. We see it in finance describing cycles in the markets.

   To get the next term in this sequence, you add the two previous terms. For example, the sixth term 5, is calculated by adding the fifth term, 3, and the fourth term, 2. In general, the n^{th} term is found by adding the $n-1^{th}$ term and the $n-2^{th}$ term. In mathematical notation, this looks like $a_n = a_{n-1} + a_{n-2}$. The next three terms will be 55, 89, and 144.

   When calculating a term in the sequence depends on prior terms in the sequence, we call this a *recursive sequence*. The idea is analogous to an *iteration* in computer programming.

Not all sequences of numbers have patterns, however (and even when they do, those patterns are not always addition or multiplication). They can be random, like the sequence of digits in the decimal expansion of π appears to be.

$$\pi = 3.14159265358979323846264338327 95 \ldots \ldots$$

If we write the digits of this decimal expansion in a sequence, it looks like this: $\{1,4,1,5,9,2,6,5,3,5,8,9,7,9,3,2,3,8,4,6,2,6,4,3,3,8,3,2,7,9,5 \ldots\}$

There is no identifiable pattern (as of yet) in this sequence. As a matter of fact, this is an open problem in mathematics and if you can find a discernible pattern, you will surely win the Field Prize in Mathematics!!