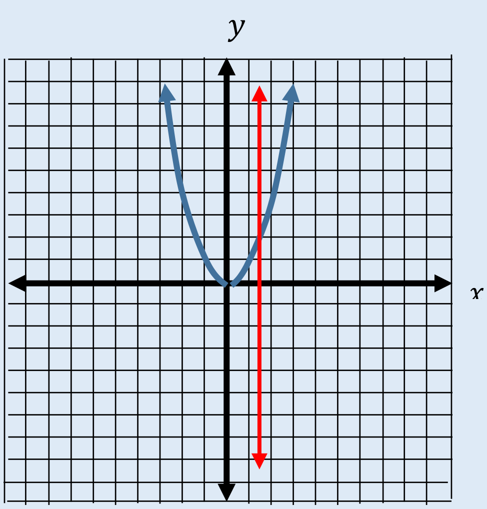
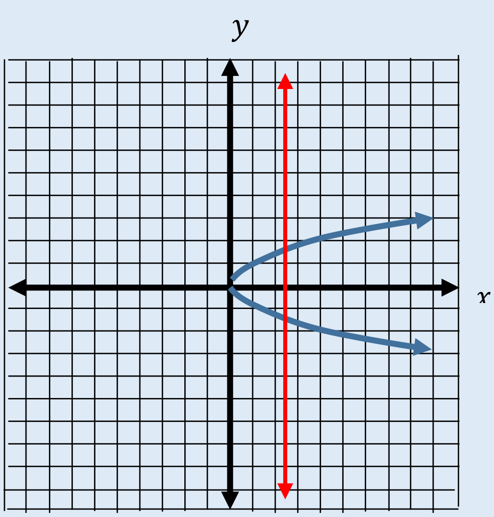
6.5 Inverse Functions

The idea of an inverse is basically something that "undoes" something else. We can think of many inverse operations, for example, like addition and subtraction (they "undo" each other) or multiplication and division (they also "undo" each other). As we will see, inverse functions are just functions that "undo" each other. They must, however, first be functions in order to be inverse functions. (If we lighten that restriction, we could consider inverse relations as well).

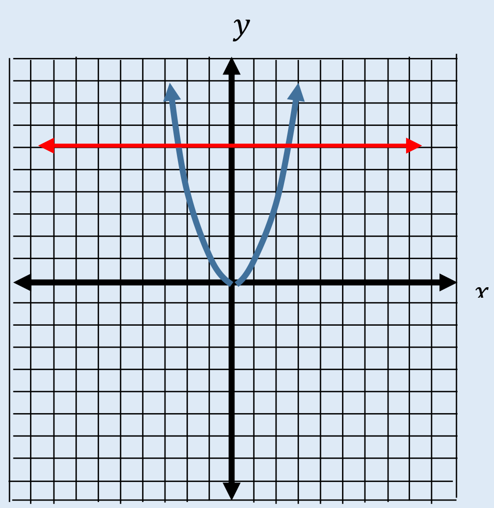
Recall that in order for a relation to be a **function**, for each x value in the domain there can be at most one y value in the range. We can tell by looking at the graph of a relation whether or not it is a function because the vertical line test reveals it when there is more than one y value for a given x value.



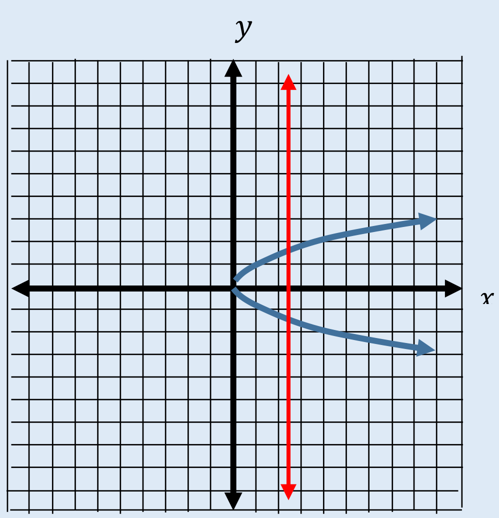


Above are the graphs of the relations $y = x^2$ and $x = y^2$. It is easy to see using the vertical line test that $y = x^2$ is a function while $x = y^2$ is not. (These are called inverse relations because the variables x and y are switched in the equations, but they are not inverse functions since one of them is not a function). To relate this to the idea of "undoing" each other, solve each for y and take a look at them again. We would have $y = x^2 \ and\ y = \pm\sqrt{x}$. Without the $\pm$ symbol (and assuming $x > 0$), it is easy to see that these are inverse operators since the square and the square root "undo" each other!

A **function** is called *one-to-one* if for each y value in the range, there is at most one x value in the domain. Keep in mind that we are defining this notion of *one-to one* on **functions** as a part of the definition. So that means that for each x value there is at most one y value (definition of a function) and for each y value, there is at most one x value. A shorter way to say this is "one x for one y", hence the name *one-to-one*. Now let's consider those graphs again. We can use the horizontal line test to tell if a function is one-to-one. To use the horizontal test, just check that every horizontal line hits the graph at most once (just like the vertical line test). Remember that any relation has to be a function in order to be one-to-one, so by default it must pass both the vertical and the horizontal line tests:

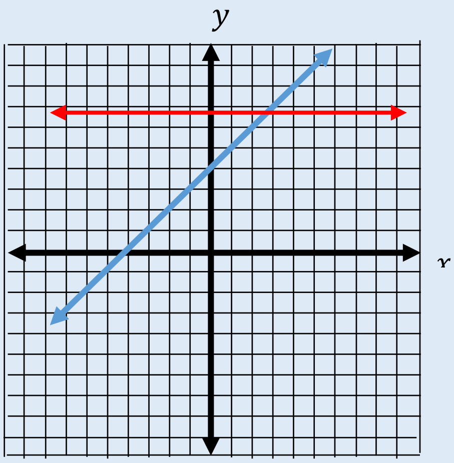


This is a function, but it is not one-to-one because it does not pass the horizontal line test.

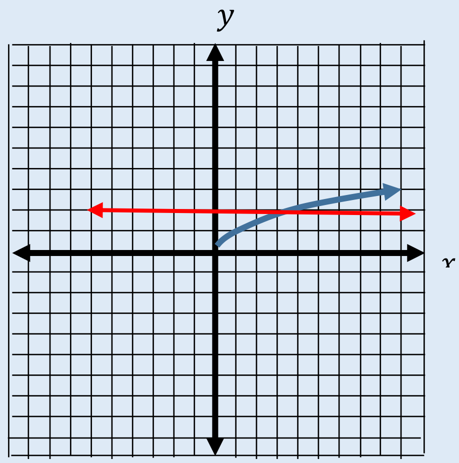


This is not even a function, so it is not one-to-one.

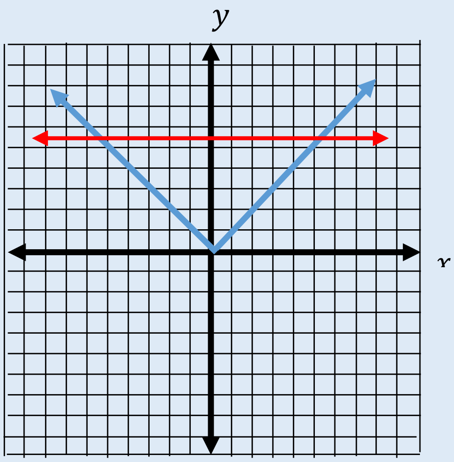
Let’s consider some more relations:



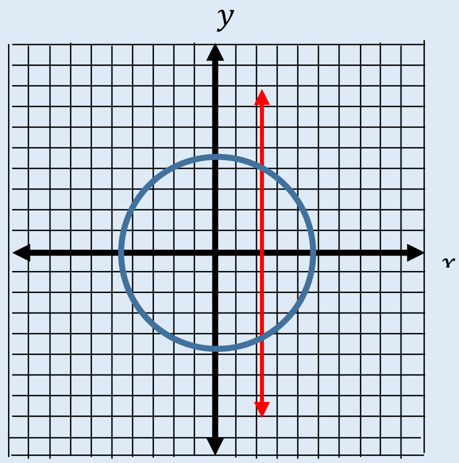
Any line is a function, so we only need to do the horizontal line test to see that it is indeed one-to-one.



The graph of $y = \sqrt{x}$, or the top half of the parabola $x = y^2$, is a function and we can see that it passes the horizontal line test. Therefore, it is one-to-one.



Here we have the absolute value function $y = |x|$, which we see does not pass the horizontal line test, so it is not one-to-one.

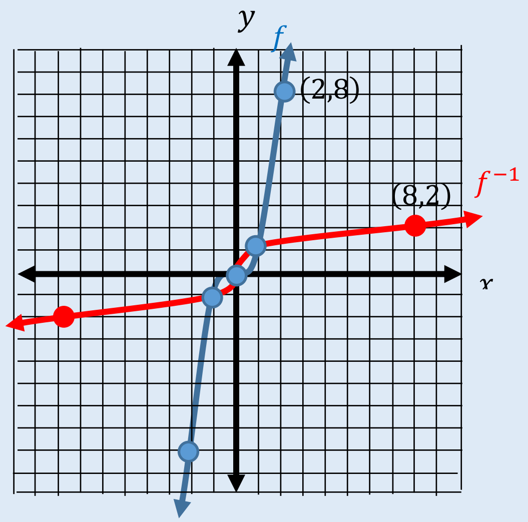


The graph of any circle $x^2 + y^2 = r^2$ is not a function so it cannot be one-to-one.

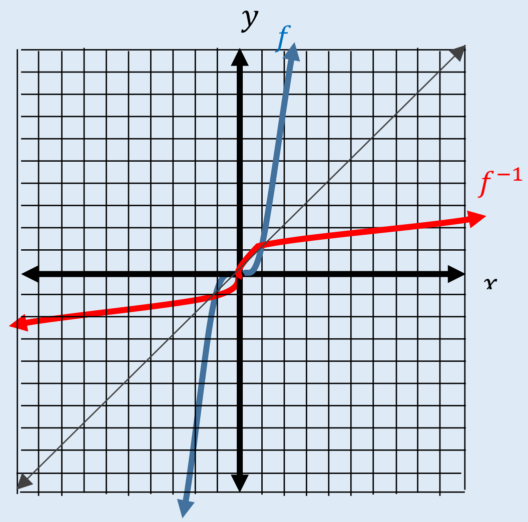
A function must be one-to-one in order to have an inverse (function). We will see toward the end of the section that we can restrict the domain of a function to find the inverse for a piece of it, however.

To find an inverse graphically, first make sure that you have a one-to-one function and then switch the positions of x and y in your points (it only takes a few points before you can see the shape). Another way is to reflect your graph about the line $y = x$, since this will automatically switch x and y in the points on the graph.

Notation alert: To denote the inverse of a function, a superscript of -1 is used, but it should not be confused with the exponent -1. For a function f or $f(x)$, to denote its inverse we write $f^{-1} or\ f^{-1}(x)$, which is read "f inverse" or "f inverse of x". Be careful because it does not mean the same thing as an exponent: $f^{-1} \neq \frac{1}{f}$.



Consider the graph of $f(x) = x^3$ which is a one-to-one function and thus does have an inverse. If we switch the values for x and y for a few points (e.g. (8,2) on f becomes (2,8) on f^{-1}), we can quickly trace out the inverse function.



Another way to visualize the inverse is to draw the line $y = x$ and use it as a mirror to reflect the graph. Either method works. Here, we use the reflection method for the same graph, $y = x^3$.

We now turn to the algebraic portion of the lesson. We begin by appealing to the intuition, noting that inverse functions "undo" each other just like inverse operations "undo" each other. This makes it easy to find the inverse for some simple functions, although we will develop a technique that will work for all one-to-one functions.

Examples

Find the inverse of the given one-to-one function by reversing the operations on the variable.

1. $f(x) = x + 1$

   The opposite of adding 1 is subtracting 1, so our intuition tells us that the inverse function should be $f^{-1}(x) = x - 1$.

2. $g(x) = 2x$

   The opposite of multiplying by 2 is dividing by 2, so our intuition tells us that the inverse function should be $g^{-1}(x) = \frac{x}{2}$.

3. $h(x) = 2x + 3$

   Here we have two operations happening, but remember that we are "undoing" here, so we should go in the reverse order of operations, first "undoing" the addition and then "undoing" the multiplication. That means we should subtract 3 and then divide by 2. This leads us to believe that our inverse function should be $h^{-1}(x) = \frac{x-3}{2}$.

We can actually see if we are correct by checking to see if they really do "undo" each other. This means when we compose the function with its inverse, we should end up back where we began, with x. In other words, if I start with a value (call it x) and then I perform a function f on it, I arrive at some other number (call it y). Now, if I perform the inverse function on y, it should take me back to x. This means that inverses always have the property that $(f^{-1} \circ f)(x) = x$ and similarly $(f \circ f^{-1})(x) = x$. This is how we can verify whether or not two functions are inverses of each other.

Examples

Show that the given functions are inverses of each other.

1. $f(x) = x + 1; \;\; g(x) = x - 1$

We must show that $(f \circ g)(x) = x$ and that $(g \circ f)(x) = x$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x - 1) \\ &= (x - 1) + 1 \\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x + 1) \\ &= (x + 1) - 1 \\ &= x\end{aligned}$$

Therefore, $g = f^{-1}$.

2. $h(x) = 2x + 3;\;\; k(x) = \frac{x-3}{2}$

We must show that $(h \circ k)(x) = x$ and that $(k \circ h)(x) = x$.

$$\begin{aligned}(h \circ k)(x) &= h(k(x)) \\ &= h\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right) + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

$$\begin{aligned}(k \circ h)(x) &= k(h(x)) \\ &= k(2x + 3) \\ &= \frac{(2x+3)-3}{2} \\ &= \frac{2x+3-3}{2} \\ &= \frac{2x}{2} \\ &= x\end{aligned}$$

Therefore, $k = h^{-1}$.

3. $f(x) = 2^x;\;\; g(x) = \log_2 x$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\log_2 x) \\ &= 2^{\log_2 x} \\ &= x\end{aligned}$$

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(2^x) \\
&= \log_2 2^x \\
&= x
\end{aligned}
$$

Therefore, $g = f^{-1}$.

In order to find the inverse of a function, we can switch the variables x and y and solve for y. The next set of examples focuses on this technique.

Examples

Find the inverse of each of the following one-to-one functions.

1. $f(x) = 2x + 3$

First, rewrite it with two variables and then switch the variables to get the inverse:

$$y = 2x + 3$$

So the inverse is:

$$x = 2y + 3$$

But it is not in the right form, so we need to solve it for y:

$$\Rightarrow \quad x - 3 = 2y$$

$$\Rightarrow \quad \frac{x-3}{2} = y$$

Now, replace the variable y with the inverse function notation $f^{-1}(x)$: $\quad \Rightarrow \quad f^{-1}(x) = \frac{x-3}{2}$

2. $g(x) = \frac{x^3+5}{2}$

First, rewrite it with two variables and then switch the variables to get the inverse:

$$y = \frac{x^3+5}{2}$$

So the inverse is:

$$x = \frac{y^3+5}{2}$$

But it is not in the right form, so we need to solve it for y:

$$\Rightarrow \quad 2x = y^3 + 5$$

$$\Rightarrow \quad 2x - 5 = y^3$$

$$\Rightarrow \quad \sqrt[3]{2x-5} = y$$

Now, replace the variable y with the inverse function notation $f^{-1}(x)$:

$$\Rightarrow \quad f^{-1}(x) = \sqrt[3]{2x-5}$$

3. $f(x) = \frac{2x+1}{3x-5}$

First, rewrite it with two variables and then switch the variables to get the inverse:

$$y = \frac{2x+1}{3x-5}$$

So the inverse is:

$$x = \frac{2y+1}{3y-5}$$

But it is not in the right form, so we need to solve it for y:

$\Rightarrow \quad x(3y-5) = 2y+1$

$\Rightarrow \quad 3xy - 5x = 2y + 1$

$\Rightarrow \quad 3xy - 2y = 5x + 1$

$\Rightarrow \quad y(3x-2) = 5x + 1$

$\Rightarrow \quad y = \frac{5x+1}{3x-2}$

We need to separate the variable here, collecting all of our $y's$ on one side so we can factor it out….

Now, replace the variable y with the inverse function notation $f^{-1}(x)$:

$\Rightarrow \quad f^{-1}(x) = \frac{5x+1}{3x-2}$

4. $y = \log_2(x-3) + 1$

This one is already written with two variables so switch the variables to get the inverse:

$$x = \log_2(y-3) + 1$$

But it is not in the right form, so we need to solve it for y:

$$\Rightarrow \quad x - 1 = \log_2(y - 3)$$

$$\Rightarrow \quad 2^{x-1} = y - 3$$

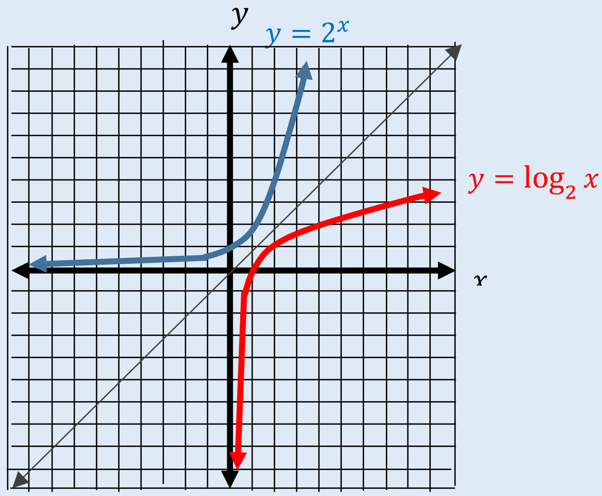
$$\Rightarrow \quad 2^{x-1} + 3 = y$$

We can use the definition of a logarithm to rewrite the log as an exponential. (Slide the base underneath the other side).

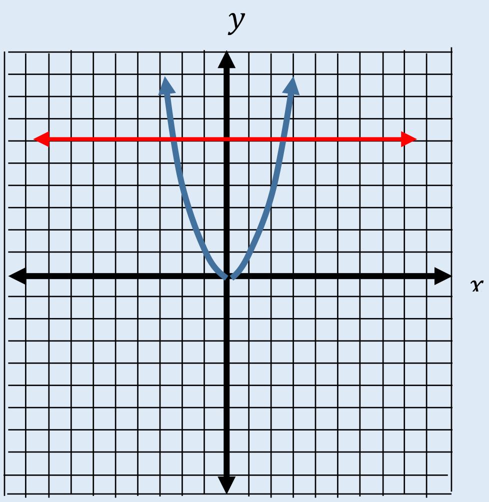
Now, replace the variable y with the inverse function notation $f^{-1}(x)$:

$$\Rightarrow \quad f^{-1}(x) = 2^{x-1} + 3$$

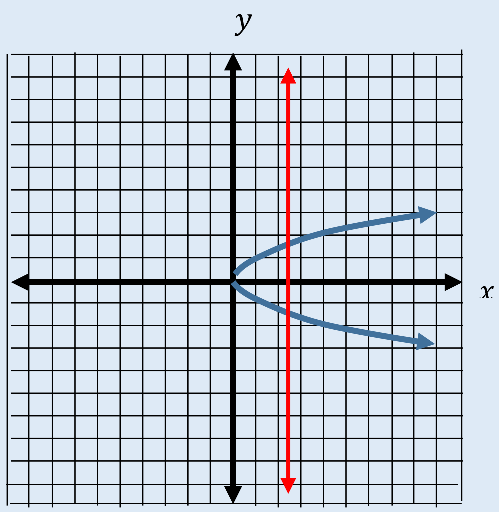
Note: It is not surprising that logarithms and exponentials are inverse functions since their operations "undo" each other. If we look at the graphs of $y = \log_2 x$ and 2^x, for example, we can see that they are mirror images about the line $y = x$:



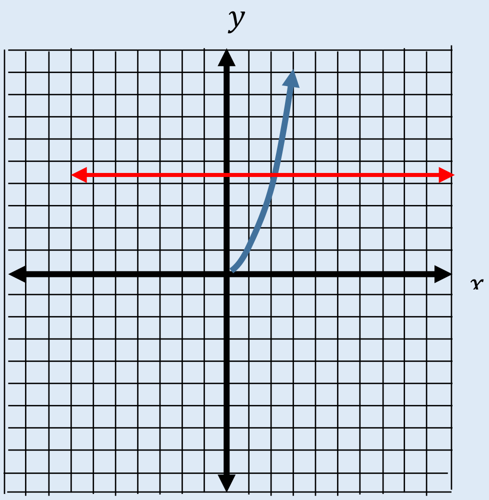
Some of our examples will have restricted domains in order to obtain a one-to-one piece of the function. Recall our first set of examples $y = x^2$ and its inverse relation $x = y^2$.



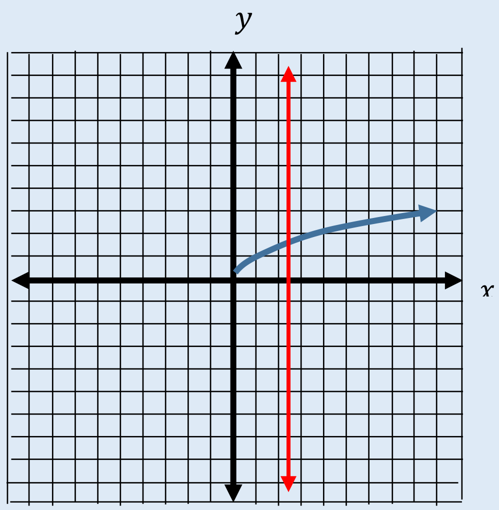
Recall that our parabola $y = x^2$ is not one-to-one because it does not pass the horizontal line test and hence it has no inverse function.



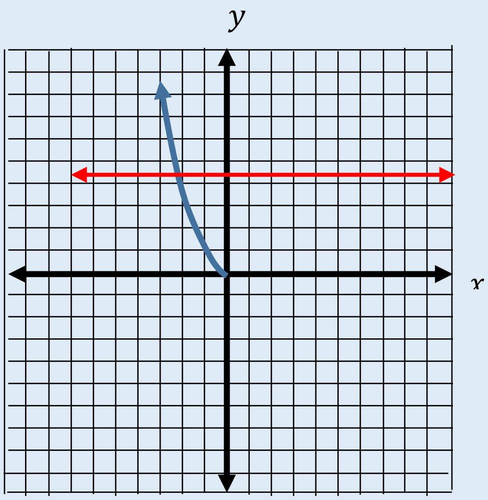
The inverse relation $x = y^2$ is not a function.

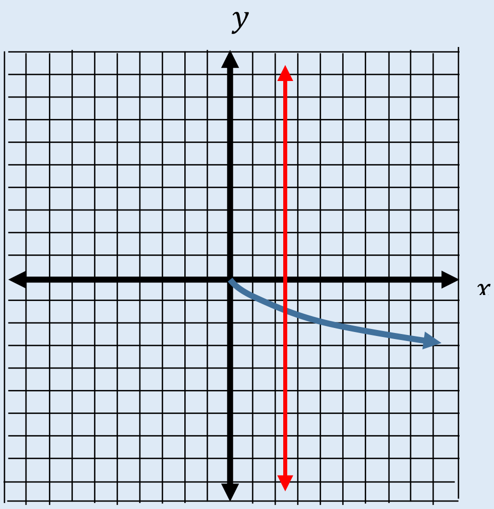


If we restrict the domain of the parabola to $x \geq 0$, then we have a one-to-one function.



The inverse of the right half of the parabola $y = x^2$ is the top half of the parabola $x = y^2$ which is just the function $y = \sqrt{x}$.





If we restrict the domain of the parabola to $x \leq 0$, then we also have a one-to one function.

The inverse of the left half of the parabola $y = x^2$ is the bottom half of the parabola $x = y^2$ which is just the function $y = -\sqrt{x}$.

By using these restrictions, we can switch the variables to find the inverse, but we need to be careful to specify the domain of the inverse when it is not clear from the form of the function.

5. $y = (x-1)^2 + 3; \;\; x \leq 1$

If you picture the graph of this function, it is the left half of the parabola, so the inverse will be the bottom half of the corresponding sideways parabola. We will use this information at the end of the problem.

Switch the variables to get the inverse:

$$x = (y-1)^2 + 3$$

Now solve it for y:

$$\Rightarrow \quad x - 3 = (y-1)^2$$

$$\Rightarrow \quad \pm\sqrt{x-3} = y - 1$$

$$\Rightarrow \quad 1 \pm \sqrt{x-3} = y$$

We will choose the negative sign since that corresponds to the bottom half of the sideways parabola.

$$\Rightarrow \quad 1 - \sqrt{x-3} = y$$

Now, replace the variable y with the inverse function notation $f^{-1}(x)$:

$$\Rightarrow \quad f^{-1}(x) = 1 - \sqrt{x-3}$$

The form of this function already indicates the range is restricted to the bottom half, so that there is not confusion.
The domain is $x \geq 3$ if we need to write it down.

6. $f(x) = 3 - \sqrt{x+2}; \;\; x \geq -2$

If you picture the graph of this function, it is the bottom half of the sideways parabola, so the inverse will be the left half of the corresponding upright parabola. We will use this information at the end of the problem to help us find the domain.

$$y = 3 - \sqrt{x+2}$$

$$x = 3 - \sqrt{y+2}$$

Now solve it for y:

$$\Rightarrow \quad x - 3 = -\sqrt{y+2}$$

$$\Rightarrow \quad (x-3)^2 = \left(-\sqrt{y+2}\right)^2$$

The negative sign gets squared as well.

$$\Rightarrow \quad (x-3)^2 = y + 2$$

$$\Rightarrow \quad (x-3)^2 - 2 = y$$

Now choose the left side of the parabola, which means we need to restrict the domain to $x \leq 3$.

Now, replace the variable y with the inverse function notation $f^{-1}(x)$:

$$\Rightarrow \quad f^{-1}(x) = (x-3)^2 - 2; \;\; x \leq 3$$

If it helps, graph the functions and their inverses to visually see the domain and range of each.