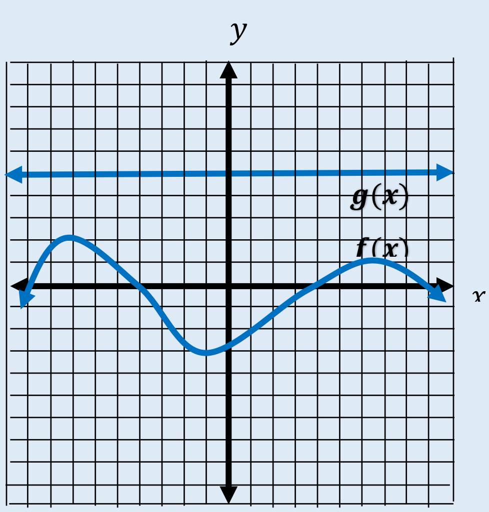
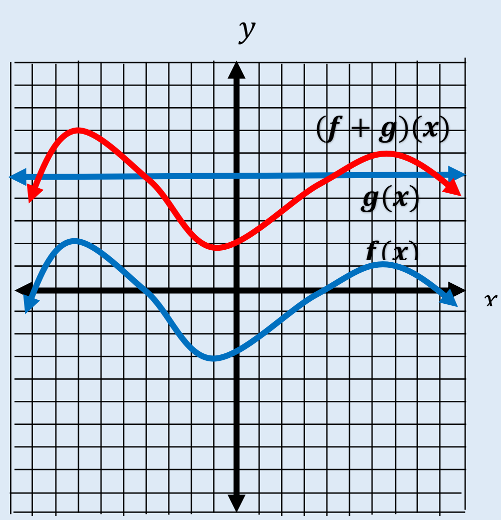
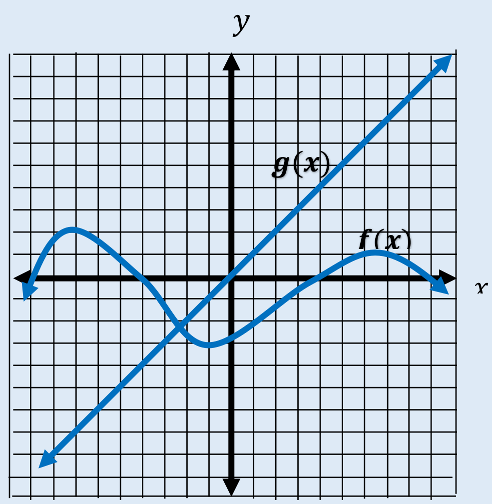
6.4 The Algebra of Functions

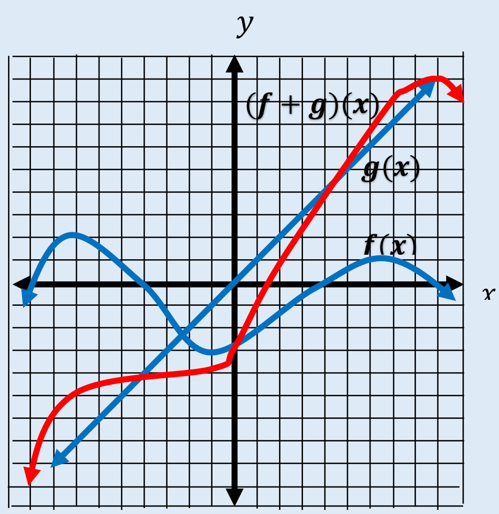
Just as we have operations with numbers (e.g. addition, subtraction, multiplication and division), we also have operations with functions. Since the output of a function is a real number, it makes sense to add, subtract, multiply and divide (except by 0) them. That is exactly what we do when we "operate" on functions – we just "operate" on the y values (the output) which are real numbers. For example, if you want to add two functions f and g together, you would just add their y values at each point to get a new function $f + g$. We can think about this visually as a shift up, just as we did when we were adding numbers to functions. The only difference here is that we might be adding a different value at each point since g can vary. It is easy to visualize when g is constant (a horizontal line) since then we are adding the same value at every point and the result is a vertical shift of f:





Here $g(x) = 5$, so adding $g(x)$ to $f(x)$ just means adding 5 to each y value of f. Try to imagine $g(x)$ being something other than a horizontal line. You would still add the y values together, but $g(x)$ would not be the same value at each point.





Here $g(x) = x$, so adding $g(x)$ to $f(x)$ means adding the x value to each y value of f. The line "smooths out" and rotates the curve when we add it to the function.

The pictures above should give us an idea of what happens graphically when we add two functions, but a similar thing happens when we subtract, multiply or divide two functions. We perform these operations "pointwise", which means at each x value, we perform the addition subtraction, multiplication, division on the associated y values called $f(x)$ and $g(x)$. The operations are defined as follows:

Addition: $(f+g)(x) = f(x) + g(x)$

Subtraction: $(f-g)(x) = f(x) - g(x)$

Multiplication: $(f \cdot g)(x) = f(x) \cdot g(x)$

Division: $\left({}^{f}\!/_{g}\right)(x) = \frac{f(x)}{g(x)}$ $for\ all\ x\ such\ that\ g(x) \neq 0$

__**Examples**__

Perform the indicated operation(s) and simplify.

1. Given $f(x) = x^2 + 3x - 1$ and $g(x) = x^2 - 2x + 5$, find $(f + g)(x)$ and $(f - g)(x)$.

$$(f + g)(x) = f(x) + g(x)$$

$$= (x^2 + 3x - 1) + (x^2 - 2x + 5)$$

$$= x^2 + 3x - 1 + x^2 - 2x + 5$$

$$= 2x^2 + x + 4$$

Note: This means that $f + g$ is a parabola, and for any value of x, we can get the y value by plugging our value for x into the expression $2x^2 + x + 4$. For example, $(f + g)(3) = 2 \cdot 3^2 + 3 + 4 = 25$, so we have the point $(3,25)$ on the graph of the function $f + g$.

$$(f - g)(x) = f(x) - g(x)$$

$$= (x^2 + 3x - 1) - (x^2 - 2x + 5)$$

$$= x^2 + 3x - 1 - x^2 + 2x - 5$$

$$= 5x - 6$$

Note: This means that $f - g$ is a line and for any value of x, we can get the y value by plugging our value for x into the expression $5x - 6$. For example, $(f - g)(-1) = 5(-1) - 6$

$= -11$, so we have the point $(-1, -11)$ on the graph of the function $f - g$.

2. Given $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{1}{x}$, find $(f + g)(x)$ and $(f - g)(x)$.

$$(f + g)(x) = f(x) + g(x)$$

$$= \frac{1}{x-2} + \frac{1}{x}$$

$$= \frac{x}{x(x-2)} + \frac{x-2}{x(x-2)}$$

$$= \frac{2x-2}{x(x-2)} \; or \; \frac{2(x-1)}{x(x-2)}$$

$$(f - g)(x) = f(x) - g(x)$$

$$= \frac{1}{x-2} - \frac{1}{x}$$

$$= \frac{x}{x(x-2)} - \frac{x-2}{x(x-2)}$$

$$= \frac{x-x+2}{x(x-2)}$$

$$= \frac{2}{x(x-2)}$$

Note: Both functions, $f + g$ and $f - g$, are undefined when $x = 0$ and $x = 2$. (Graphically, it means that you have vertical asymptotes at these values, but we are not graphing rational functions in this course, so you will explore this more in the next course.) The domain for each of these functions is "all real numbers except 0 and 2" or in interval notation: $(-\infty, 0) \cup (0,2) \cup (2, \infty)$.

For example 2, if you had been asked to evaluate $f + g$ (or $f - g$) at a given value, it would be easiest to plug the value in to each function f and g and then perform the operation, although you could find the general function first and then plug in the value. It is up to you, but we will show you both ways here. Let's evaluate $f + g$ at the value $x = 4$ to illustrate:

By evaluating each function and then adding:

$$(f+g)(4) = f(4) + g(4)$$

$$= \frac{1}{4-2} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= \frac{2}{4} + \frac{1}{4}$$

$$= \frac{3}{4}$$

By adding the functions and then evaluating:

$$(f+g)(x) = f(x) + g(x)$$

$$= \frac{1}{x-2} + \frac{1}{x}$$

$$= \frac{x}{x(x-2)} + \frac{x-2}{x(x-2)}$$

$$= \frac{2x-2}{x(x-2)} \; or \; \frac{2(x-1)}{x(x-2)}$$

$$(f+g)(4) = \frac{2 \cdot 4 - 2}{4(4-2)} = \frac{8-2}{4 \cdot 2} = \frac{6}{8} = \frac{3}{4}$$

We get the same answer whether we plug in the value to each function and then add those values together or we find the general function by adding the functions first and then plugging in our value to the general function. This is true for every operation, not just addition.

Let's consider some examples with the other two operations, multiplication and division:

3. Given $f(x) = 2x + 1$ and $g(x) = 3x - 4$, find $(f \cdot g)(x)$ and $\left({}^{f}/_{g}\right)(x)$.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$= (2x+1)(3x-4)$$

$$= 6x^2 - 8x + 3x - 4$$

$$= 6x^2 - 5x - 4$$

So, if we multiply two lines together, we get a parabola! Here, the domain of $f \cdot g$ is "all real numbers" or $(-\infty, \infty)$.

$$\left({}^{f}/_{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{2x+1}{3x-4}$$

Here, we must specify the domain of ${}^{f}/_{g}$ since the division is only true for a non-zero denominator. Setting $3x - 4 \neq 0$, we get $x \neq \frac{4}{3}$ which means the domain is "all real numbers except $\frac{4}{3}$" or $\left(-\infty, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$.

4. Given $h(t) = 3t^2 - 1$ and $g(t) = 8t + 1$, find $(h \cdot g)(t)$ and $\left({}^{h}/_{g}\right)(t)$.

$$(h \cdot g)(t) = h(t) \cdot g(t)$$

$$= (3t^2 - 1)(8t + 1)$$

$$= 24t^3 + 3t^2 - 8t - 1$$

Here, the domain of $h \cdot g$ is "all real numbers" or $(-\infty, \infty)$ since the domain of any polynomial will be such.

$$\left({}^{h}/_{g}\right)(t) = \frac{h(t)}{g(t)}$$

$$= \frac{3t^2-1}{8t+1}$$

Setting $8t + 1 \neq 0$, we get $t \neq -\frac{1}{8}$ which means the domain is $\left(-\infty, -\frac{1}{8}\right) \cup \left(-\frac{1}{8}, \infty\right)$.

Now, let's put this all together in the next couple of examples:

5. Given $f(x) = \frac{1}{x}$ and $g(x) = -x$, find $(f+g)(x)$, $(f-g)(x), (f \cdot g)(x), and \ \left({}^{f}/_{g}\right)(x)$.

$$(f+g)(x) = f(x) + g(x)$$

$$= \frac{1}{x} + (-x)$$

$$= \frac{1}{x} - x$$

$$= \frac{1}{x} - \frac{x}{1}$$

$$= \frac{1}{x} - \frac{x^2}{x}$$

$$= \frac{1-x^2}{x}$$

The domain of $f+g$ is $(-\infty, 0) \cup (0, \infty)$.

$$(f-g)(x) = f(x) - g(x)$$

$$= \frac{1}{x} - (-x)$$

$$= \frac{1}{x} + x$$

$$= \frac{1}{x} + \frac{x}{1}$$

$$= \frac{1}{x} + \frac{x^2}{x}$$

$$= \frac{1+x^2}{x}$$

The domain of $f - g$ is also $(-\infty, 0) \cup (0, \infty)$.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$= \frac{1}{x}(-x)$$

$$= -1$$

Here, the domain of $f \cdot g$ is not "all real numbers" as you would probably believe initially. The reason is that we have to take into account the domains of the original functions as well. Since f is not defined at 0, we cannot have $f \cdot g$ defined at 0 either. So the domain is $(-\infty, 0) \cup (0, \infty)$. (The graph of $f \cdot g$ would look like the horizontal line $y = -1$, except there would be an open hole at the value $x = 0$.)

So, if either function has a restricted domain, then after operating on those functions, those restrictions will still apply.

$$\left({}^{f}\!/_{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{\frac{1}{x}}{-x}$$

$$= \frac{\frac{1}{x}}{-\frac{x}{1}}$$

$$= \frac{1}{x}\left(-\frac{1}{x}\right)$$

$$= -\frac{1}{x^2}$$

The domain is once again $(-\infty, 0) \cup (0, \infty)$.

6. Given $h(r) = r^2 + r$ and $f(r) = r + 1$, find $(h + f)(5)$, and $\left({}^{h}\!/_{f}\right)(r)$.

$$(h + f)(5) = h(5) + f(5)$$

$$= (5^2 + 5) + (5 + 1)$$

$$= 30 + 6 = 36$$

$$\left(h/_{f}\right)(r) = \frac{h(r)}{f(r)}$$

$$= \frac{r^2+r}{r+1}$$

We will simplify this now, but keep in mind that $f(r) \neq 0$ even if this simplifies, so when we write down the domain, we must exclude -1.

$$= \frac{r(r+1)}{r+1}$$

$= r$ (for every value of r except -1)

The domain is thus $(-\infty, -1) \cup (-1, \infty)$.

Composition of Functions

Along with the operations of addition, subtraction, multiplication and division, there is another operation defined on functions called *composition* which allows us to take results from one function and plug them into another function, thus *composing* our functions. The notation for composition is a hollow circle $\circ$, not to be confused with multiplication which is a solid circle or a dot.

Composition: $f \circ g$
Multiplication: $f \cdot g$

But what does composition look like mathematically?

$$(f \circ g)(x) = f(g(x))$$

Composition is defined to be the result of plugging one function into another function. In other words, replacing the variable in the outer function with the entire inner function (and then simplifying the result, of course).

Let's consider some examples to see how this works.

Examples

1. Given $f(x) = x^2 - 3x$ and $g(x) = 2x - 1$, find the functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

Keep in mind that the inner function plugs into the outer function and work from the inside out:

$$(f \circ g)(x) = f(g(x))$$

There are many parentheses here, but just focus on the inside first, replacing the inner function with its definition.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x - 1)\end{aligned}$$

Now take the expression inside of the parentheses and **replace** the variable in the outer function:

$$= (2x - 1)^2 - 3(2x - 1)$$

Now all we need to do is simplify:

$$= 4x^2 - 4x + 1 - 6x + 3)$$

$$= 4x^2 - 10x + 4$$

Sometimes students get confused because they see $x's$ all over the place and they are tempted to multiply x by the function instead of **replacing** x with the function. If it helps, leave a blank spot where the variable goes, and the put the inner function in its place:

$$f(\quad) = (\quad)^2 - 3(\quad)$$

Now, replace (or fill in) the blank spots with $2x - 1$:

$$f(2x-1) = (2x-1)^2 - 3(2x-1)$$

And the rest is just simplifying…

Let's look at $g \circ f$ as well:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 3x)\end{aligned}$$

Now take the expression inside of the parentheses and **replace** the variable in the outer function:

$$= 2(x^2 - 3x) - 1$$

Now all we need to do is simplify:

$$= 2x^2 - 6x - 1$$

Composition of functions is often seen in computer science applications or in any situation where multiple actions need to take place on a given variable (or number) in a particular order. Encryption (encoding for security) is one application where you will see composition of functions employed. The more layers, or functions, that are being composed, the better the encryption will be.

Back to our examples….

2. Given $f(x) = \sqrt{x}$ and $g(x) = 4x^2 - 4$, find the functions $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g)(x) = f(g(x))$$

$$= f(4x^2 - 4)$$

$$= \sqrt{4x^2 - 4}$$

Now all we need to do is simplify:

$$= \sqrt{4(x^2 - 1)}$$

$$= 2\sqrt{x^2 - 1}$$

Let's look at $g \circ f$ as well:

$$(g \circ f)(x) = g(f(x))$$

$$= g(\sqrt{x})$$

$$= 4\left(\sqrt{x}\right)^2 - 4$$

Now all we need to do is simplify:

$$= 4x - 4$$

If we want to just evaluate a composed function at a given value, it is actually easier to just plug that number into the inner function and get a number out and then take this result and plug it into the outer function. You can do it either way, however. For example, if you wish to evaluate $(f \circ g)(3)$, you could just evaluate $g(3)$ and then plug your result into f as follows:

$$\begin{aligned}(f \circ g)(3) &= f(g(3)) \\ &= f(4 \cdot 3^2 - 4) \\ &= f(32) \\ &= \sqrt{32} = 4\sqrt{2}\end{aligned}$$

Or you could go through the process of finding the general function $(f \circ g)(x) = 2\sqrt{x^2 - 1}$ as we did above and then plug in the value:

$$\begin{aligned}(f \circ g)(3) &= 2\sqrt{3^2 - 1} \\ &= 2\sqrt{8} \\ &= 2 \cdot 2\sqrt{2} \\ &= 4\sqrt{2}\end{aligned}$$

Either way, you will get the same answer.

3. Given $f(t) = \frac{1}{t^2+1}$ and $g(t) = \frac{2}{t}$, find the functions $(f \circ g)(t)$ and $(g \circ f)(t)$.

$$(f \circ g)(t) = f(g(t))$$

$$(f \circ g)(x) = f(g(t))$$

$$= f\left(\frac{2}{t}\right)$$

$$= \frac{1}{\left(\frac{2}{t}\right)^2 + 1}$$

Now all we need to do is simplify:

$$= \frac{1}{\frac{4}{t^2}+1}$$

$$= \frac{1}{\frac{4}{t^2}+\frac{t^2}{t^2}}$$

$$= \frac{1}{\frac{4+t^2}{t^2}}$$

$$= \frac{t^2}{4+t^2}$$

Taking into account the domain restrictions of f and g, we have to specify the restriction that $t \neq 0$ for our composed function.

Let's look at $g \circ f$ as well:

$$(g \circ f)(t) = g(f(t))$$

$$= g\left(\frac{1}{t^2+1}\right)$$

$$= \frac{2}{\frac{1}{t^2+1}}$$

Now all we need to do is simplify:

$$= 2(t^2 + 1) \text{ or } 2t^2 + 2$$

Again, noting the domain restrictions of f and g, we have to specify the restriction that $t \neq 0$ for our composed function.

In any calculus course, you will use the concept of composition to help you define the *derivative (or rate of change)* of a function and the algebra in those problems will look like the algebra in the following example.

4. Given the function $f(x) = -3x^2 + 2x - 1$, find the *difference quotient* $\frac{f(x+h)-f(x)}{h}$.

The composition occurs in the first piece of the numerator:

$$f(x + h) = -3(x + h)^2 + 2(x + h) - 1$$

Now, simplify this expression:

$$= -3(x^2 + 2xh + h^2) + 2x + 2h - 1$$

$$= -3x^2 - 6xh - 3h^2 + 2x + 2h - 1$$

This is as simplified as we can make it, but now plug it into the numerator and you see terms disappear:

$$\frac{f(x+h)-f(x)}{h} = \frac{-3x^2-6xh-3h^2+2x+2h-1-\left(-3x^2+2x-1\right)}{h}$$

$$= \frac{-3x^2-6xh-3h^2+2x+2h-1+3x^2-2x+1}{h}$$

$$= \frac{-6xh-3h^2+2h}{h}$$

Now, we can factor h out of the numerator and cancel it with the denominator:

$$= \frac{h(-6x-3h+2)}{h}$$

$$= -6x - 3h + 2$$

So this is the algebra involved in the calculus problem, and the rest you will learn when you get there!