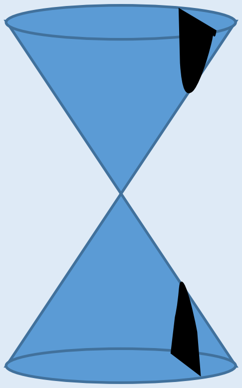
6.3 Graphing Hyperbolas and Recognizing Conics



Recall from the last section that the conics can be thought of as cross sections of a double cone. A vertical cut through the cone will give us a hyperbola, which looks similar to two parabolas.

<u>Hyperbolas</u>

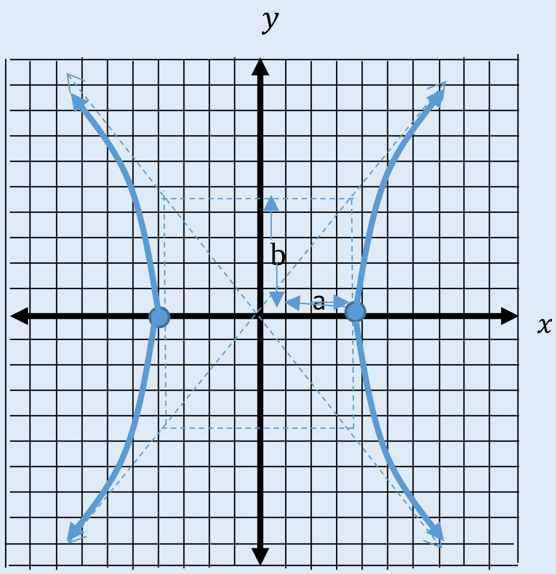
In form, the hyperbola is very similar to the ellipse. In fact, the only difference in the graphing form of the equation is the connecting sign:

Ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ if it opens side to side

or $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ if it opens up and down

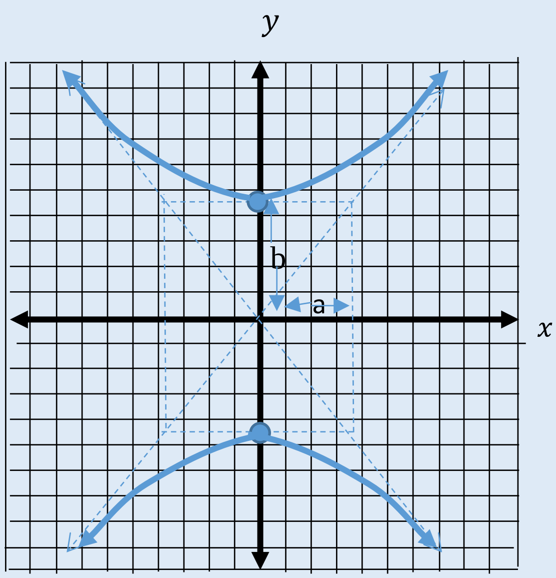
The process of graphing is somewhat similar. We still begin at the center and move out a distance of a in the horizontal direction and a distance of b in the vertical direction. Instead of connecting those points to form an ellipse, we form a rectangle which helps us get the asymptotes of the graph. Then we graph the hyperbola by approaching those asymptotes and noting whether the hyperbola opens side to side or up and down.



A hyperbola centered at the origin $(0,0)$ is pictured to the left.

The equation of this hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Note: The dashed rectangle and asymptotes in the background are not part of the actual hyperbola.



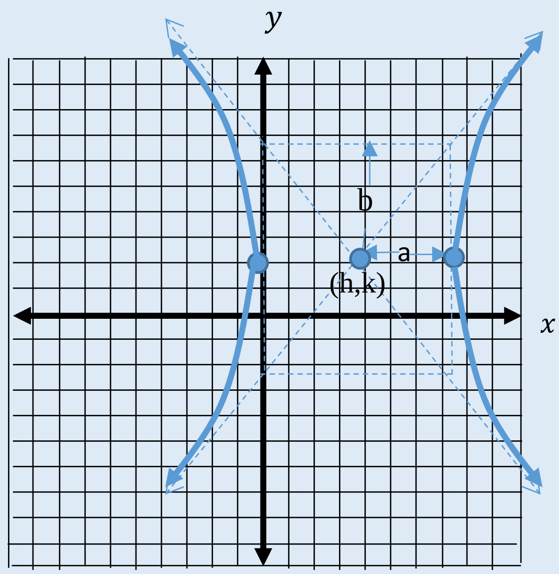
Another hyperbola centered at the origin $(0,0)$ is pictured to the left.

The equation of this hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$

Notice that y comes first (is positive) here and it opens in the y direction.

The points on the rectangle where the hyperbola actually touches are called *vertices*. The vertices for the first hyperbola are $(a, 0)$ and $(-a, 0)$. The vertices for the second parabola that opens up and down are *(0,b)* and *(0,-b)*. Even when the center is not the origin, these vertices are fairly simple to find since you move out from the center to get there by a distance of either a or b. For this reason, if your hyperbola opens side to side, the vertices will be $(h + a, 0)$ and $(h - a, 0)$. Similarly, if your hyperbola opens up and down, the vertices will be

$(0, k+b)$ and $(0, k-b)$. You will graph from your vertices toward your asymptotes being careful not to cross them. We are just *sketching* the graphs, so they are not going to be perfect, but if you plug in a few points, you will get more accurate pictures.

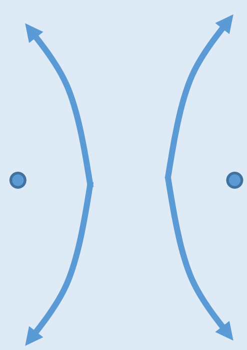


A hyperbola centered at the point (h, k) is pictured to the left.

The equation of this hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

Note: The center (h, k) is not part of the hyperbola, but a beginning place to get the asymptotes. The hyperbola is made up of the two curved pieces.

Hyperbolas, like ellipses, have foci. Their foci lie inside of each curved piece, the same distance away from each of the vertices.



Neat stuff about foci:

1. The distance, c, from the focus to a vertex can be found by using the Pythagorean Theorem $a^2 + b^2 = c^2$ where a and b are taken from the equation.

2. Foci can be used to collect and reflect sound or light waves.

Let's go through the process of graphing hyperbolas with some examples.

<u>Examples</u>

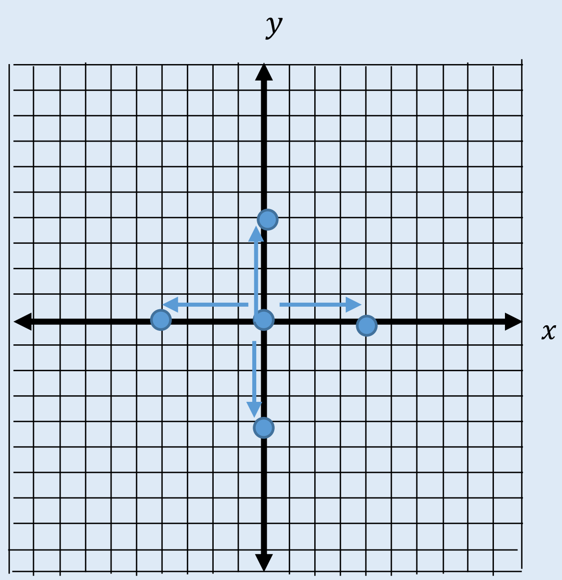
Graph each of the following relations and give the domain, range, x-intercept(s) and y-intercept(s).

1. $\frac{y^2}{16} - \frac{x^2}{16} = 1$

This hyperbola is already in graphing form with center $(0,0)$.
Also, $a = 4$ and $b = 4$ since $\sqrt{16} = 4$.

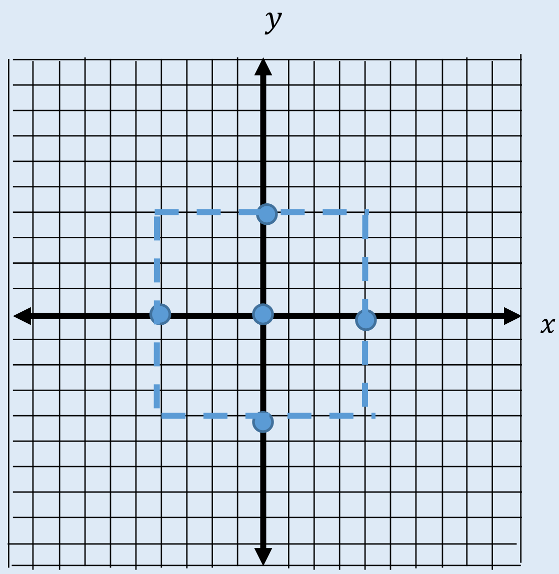
Note that the equation could be written in the following form:

$$\frac{(y-0)^2}{4^2} - \frac{(x-0)^2}{4^2} = 1$$

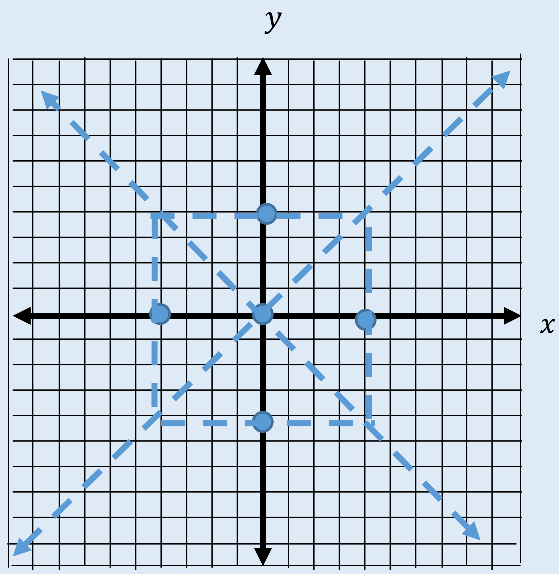


We will begin at the center $(0,0)$ and move a distance of 4 in the horizontal direction and a distance of 4 in the vertical direction to get 4 points on the rectangle.

Now trace out the rectangle by drawing horizontal and vertical lines through the four outer points.

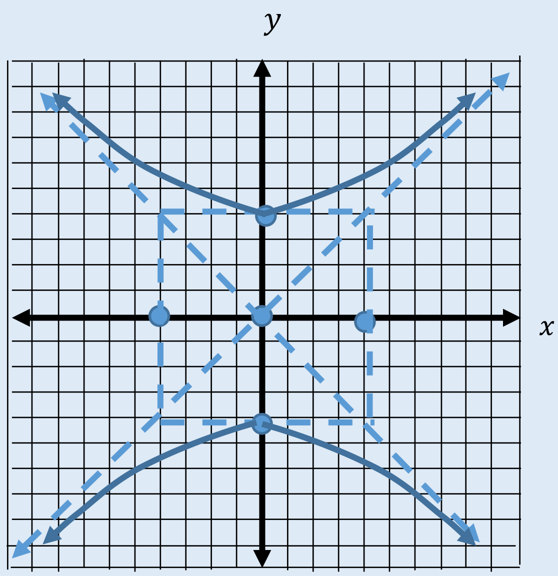


Next, draw the asymptotes by connecting the center to the corners of the rectangle.

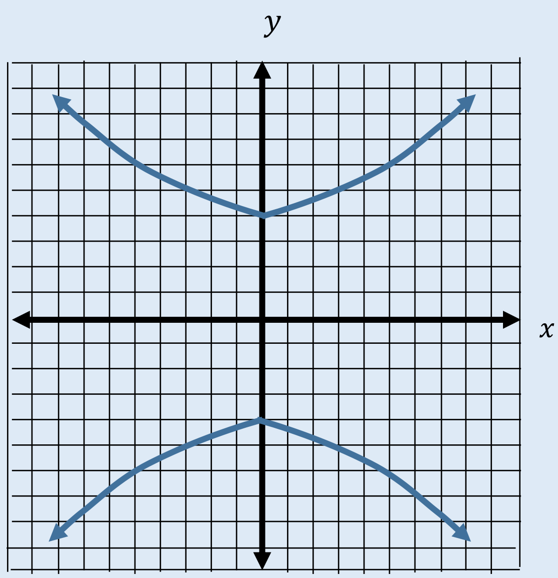


Next, decide which way the hyperbola opens – up and down or side to side? This one opens up and down because the equation has a positive y^2 term (and a negative x^2 term).

Hence the vertices are $(0,4)$ and $(0,-4)$. We can sketch the graph by starting at the vertices and approaching the asymptotes.



Now, we can remove the diagram that we used to get the graph if we like.



The domain and range are easy to read off of the graph, but you can use the vertices as well if that helps.

domain: $(-\infty, \infty)$

range: $(-\infty, -4] \cup [4, \infty)$

The intercepts are also easy to read off of the graph since this one is centered at the origin:

x-intercept(s): none

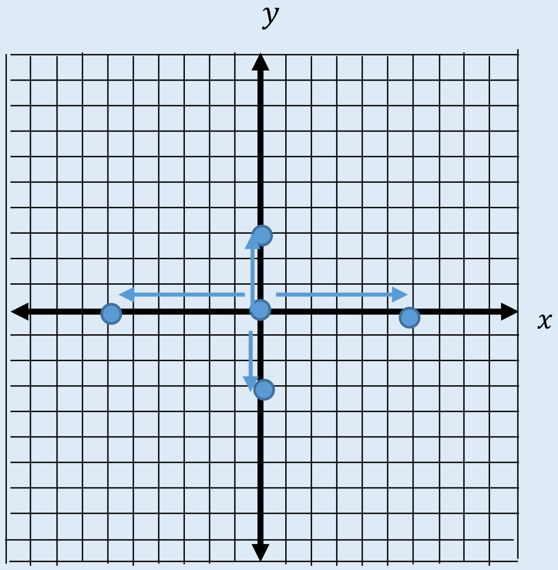
y-intercept(s): $(0,4), (0, -4)$

2. $\frac{x^2}{36} - \frac{y^2}{9} = 1$

This hyperbola is already in graphing form with center $(0,0)$. Also, $a = 6$ and $b = 3$.

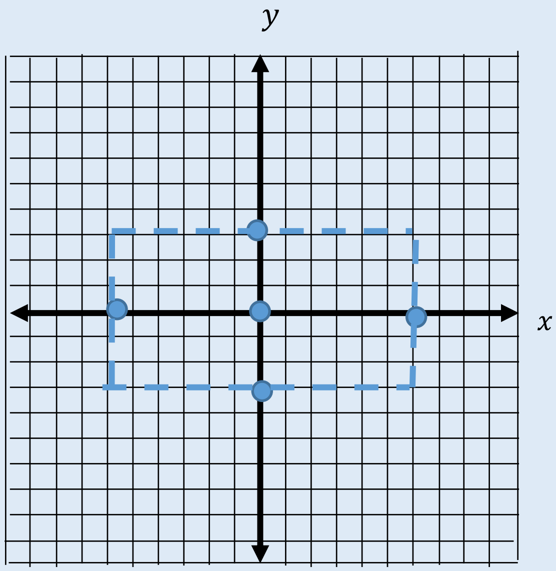
Note that the equation could be written in the following form:

$$\frac{(x-0)^2}{6^2} - \frac{(y-0)^2}{3^2} = 1$$

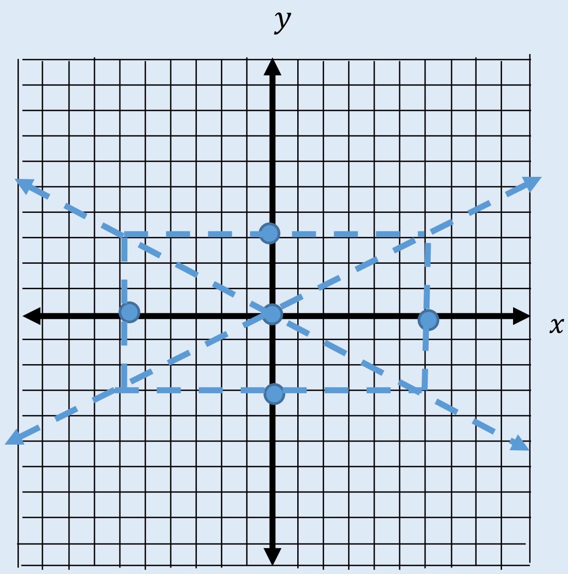


We will begin at the center $(0,0)$ and move a distance of 6 in the horizontal direction and a distance of 3 in the vertical direction to get 4 points on the rectangle.

Now trace out the rectangle by drawing horizontal and vertical lines through the four outer points.

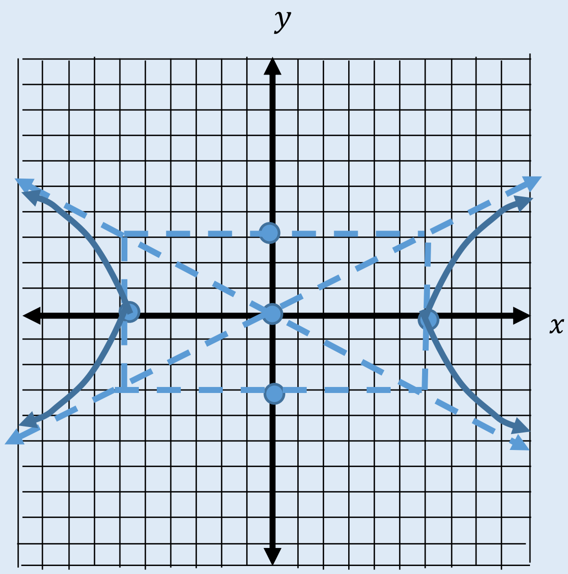


Next, draw the asymptotes by connecting the center to the corners of the rectangle.



Next, decide which way the hyperbola opens – up and down or side to side? This one opens side to side because the equation has a positive x^2 term (and a negative y^2 term).

Hence the vertices are $(6,0)$ and $(-6,0)$. We can sketch the graph by starting at the vertices and approaching the asymptotes.



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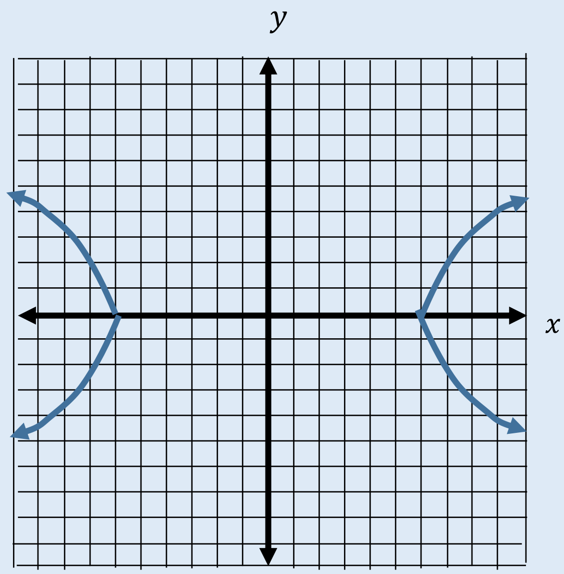


The domain and range are easy to read off of the graph, but you can use the vertices as well if that helps.

domain: $(-\infty,-6] \cup [6,\infty)$

range: $(-\infty,\infty)$

The intercepts are also easy to read off of the graph since this one is centered at the origin:

x-intercept(s): $(6,0), (-6,0)$

y-intercept(s): none

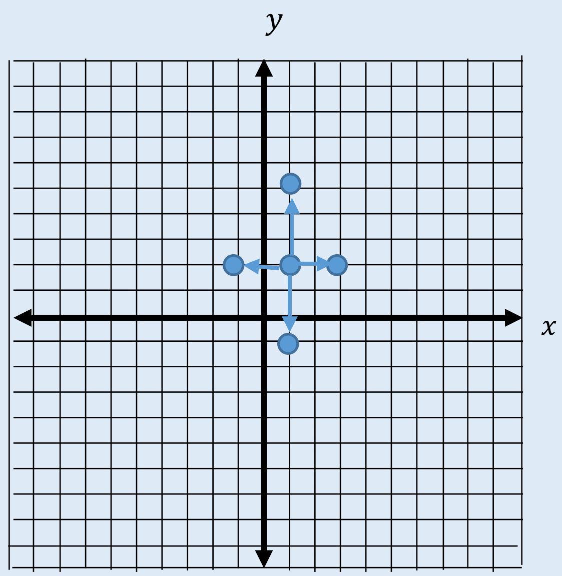
We will look at some examples that are centered elsewhere next.

3. $\frac{(y-2)^2}{9} - \frac{(x-1)^2}{4} = 1$

This hyperbola is already in graphing form with center $(1,2)$. Also, $a = 2$ and $b = 3$.

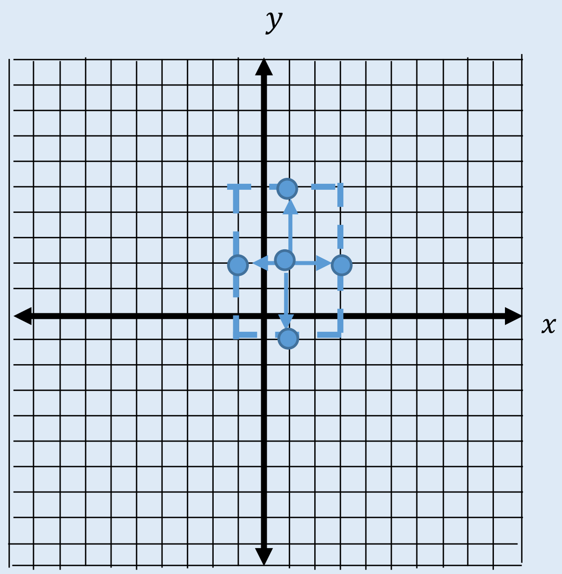
Note that the equation could be written in the following form:

$$\frac{(y-2)^2}{3^2} - \frac{(x-1)^2}{2^2} = 1$$

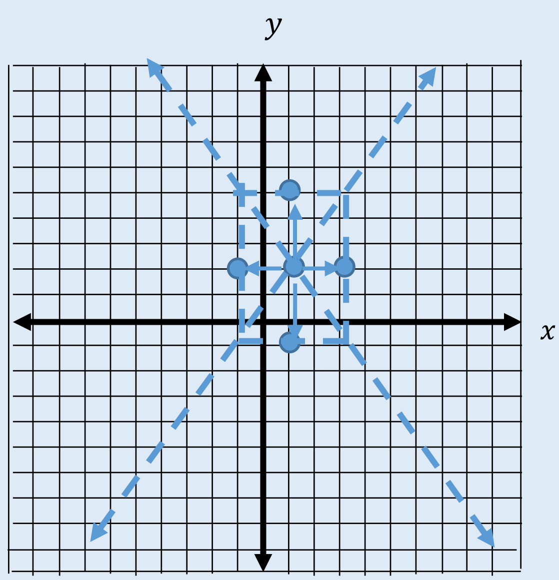


We will begin at the center $(1,2)$ and move a distance of 2 in the horizontal direction and a distance of 3 in the vertical direction to get 4 points on the rectangle.

Now trace out the rectangle by drawing horizontal and vertical lines through the four outer points.



Next, draw the asymptotes by connecting the center to the corners of the rectangle.



Note: We could find the equations of these asymptotes using the point slope formula $y - y_1 = m(x - x_1)$ since we can use the center as our point and the slopes are clearly $\pm\frac{b}{a}$ from the picture (measure $\frac{rise}{run}$ from the center to a corner of the rectangle).

Asymptote 1: $y - y_1 = m(x - x_1)$

⇨ $y - 2 = \frac{3}{2}(x - 1)$

⇨ $y = \frac{3}{2}x + \frac{1}{2}$

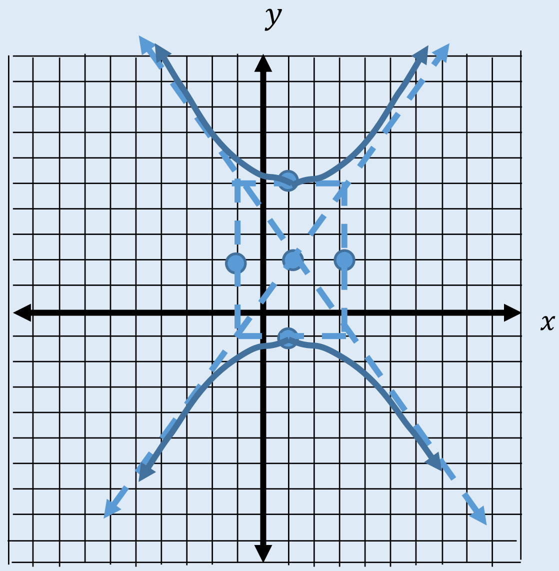
Asymptote 2: $y - y_1 = m(x - x_1)$

⇨ $y - 2 = -\frac{3}{2}(x - 1)$

⇨ $y = -\frac{3}{2}x + \frac{7}{2}$

Next, decide which way the hyperbola opens – up and down or side to side? This one opens up and down because the equation has a positive y^2 term (and a negative x^2 term).

Hence the vertices are $(2,4)$ and $(2,-2)$. We can sketch the graph by starting at the vertices and approaching the asymptotes.



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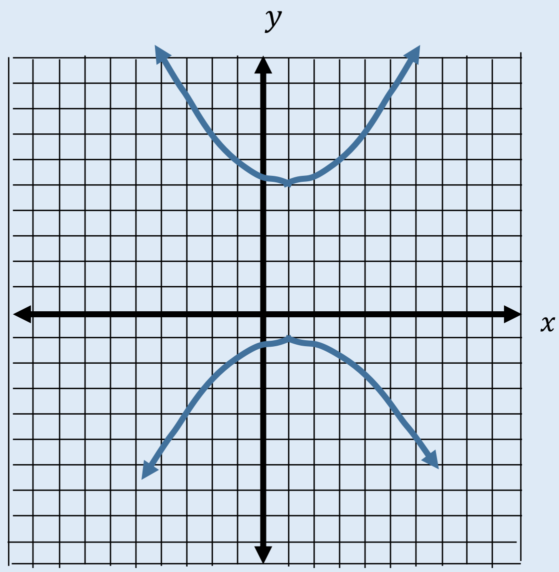


The domain and range are easy to read off of the graph, but you can use the vertices as well if that helps.

domain: $(-\infty, \infty)$

range: $(-\infty, -2] \cup [4, \infty)$

The intercepts may require a bit of work since this one is not centered at the origin. We can see that there are no x intercepts, but there are two y-intercepts. To find the y-intercepts, plug in 0 for x and solve:

$$\frac{(y-2)^2}{9} - \frac{(0-1)^2}{4} = 1$$

$$\Rightarrow \quad \frac{(y-2)^2}{9} - \frac{1}{4} = 1$$

$$+\frac{1}{4} \quad +\frac{1}{4}$$

$$\Rightarrow \quad \frac{(y-2)^2}{9} = \frac{5}{4}$$

$$1 + \frac{1}{4} = \frac{4}{4} + \frac{1}{4} = \frac{5}{4}$$

$$\Rightarrow \quad \sqrt{\frac{(y-2)^2}{9}} = \pm\sqrt{\frac{5}{4}}$$

$$\Rightarrow \quad \frac{y-2}{3} = \pm\frac{\sqrt{5}}{2}$$

$$\Rightarrow \quad y - 2 = \pm\frac{3\sqrt{5}}{2}$$

$$\Rightarrow \quad y = 2 \pm \frac{3\sqrt{5}}{2}$$

y-intercept(s): $\left(0,2+\frac{3\sqrt{5}}{2}\right), \left(0,2-\frac{3\sqrt{5}}{2}\right)$

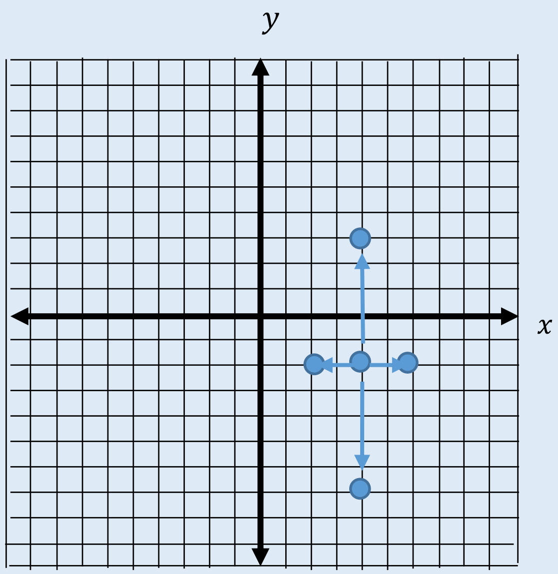
or $\approx$ $(0,5.4), (0,-1.4)$

4. $\frac{(x-4)^2}{4} - \frac{(y+2)^2}{25} = 1$

This hyperbola is already in graphing form with center $(4,-2)$. Also, $a = 2$ and $b = 5$.

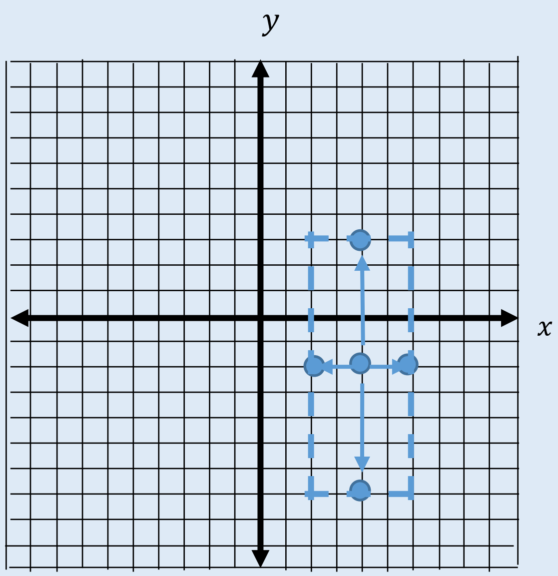
Note that the equation could be written in the following form:

$$\frac{(x-4)^2}{2^2} - \frac{(y+2)^2}{5^2} = 1$$

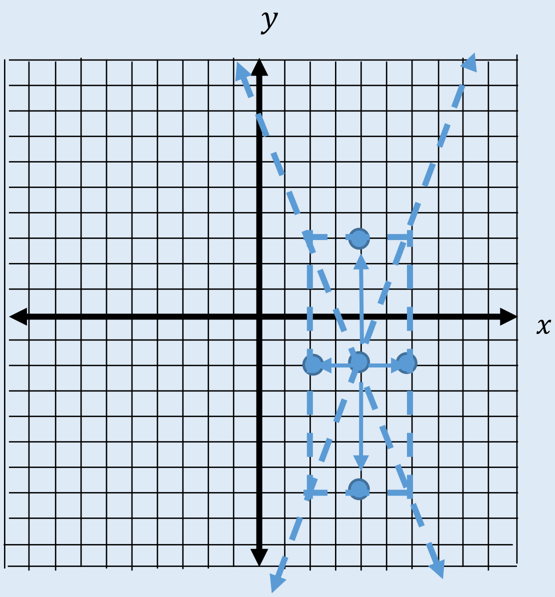


We will begin at the center $(4, -2)$ and move a distance of 2 in the horizontal direction and a distance of 5 in the vertical direction to get 4 points on the rectangle.

Now trace out the rectangle by drawing horizontal and vertical lines through the four outer points.



Next, draw the asymptotes by connecting the center to the corners of the rectangle.



Note: We could once again find the equations of these asymptotes using the point slope formula $y - y_1 = m(x - x_1)$:

Asymptote 1: $y - y_1 = m(x - x_1)$

⇨ $y - (-2) = \frac{5}{2}(x - 4)$

⇨ $y + 2 = \frac{5}{2}x - 10$

⇨ $y = \frac{5}{2}x - 12$

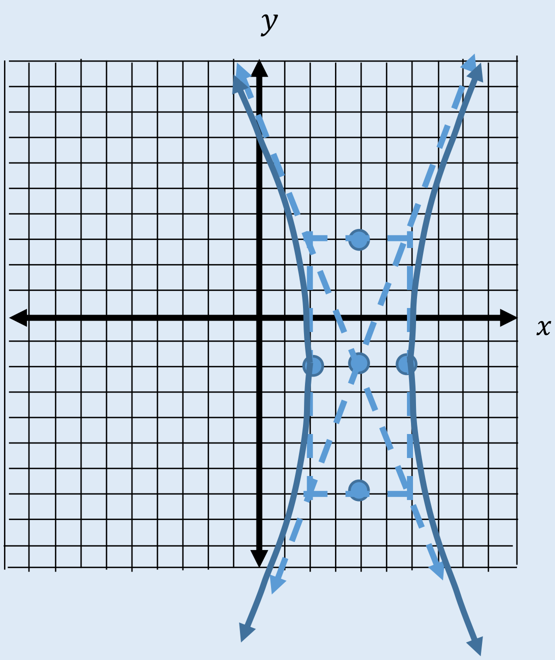
Asymptote 2: $y - y_1 = m(x - x_1)$

⇨ $y - (-2) = -\frac{5}{2}(x - 4)$

⇨ $y + 2 = -\frac{5}{2}x + 10$

⇨ $y = -\frac{5}{2}x + 8$

Next, decide which way the hyperbola opens – up and down or side to side? This one opens side to side because the equation has a positive x^2 term (and a negative y^2 term). Hence the vertices are $(2, -2)$ and $(6, -2)$. We can sketch the graph by starting at the vertices and approaching the asymptotes.



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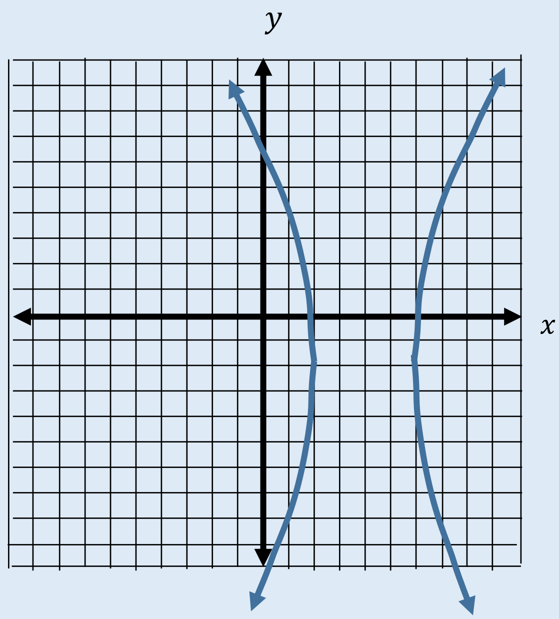

The domain and range are easy to read off of the graph, using the vertices as endpoints:

domain: $(-\infty, 2] \cup [6, \infty)$

range: $(-\infty, \infty)$

The intercepts may require a bit of work since this one is not centered at the origin. We can see that there are two x intercepts, and two y-intercepts. To find the x-intercepts, plug in 0 for y and solve:

$$\frac{(x-4)^2}{4} - \frac{(0+2)^2}{25} = 1$$

$$\Rightarrow \quad \frac{(x-4)^2}{4} - \frac{4}{25} = 1$$

$$+\frac{4}{25} \qquad +\frac{4}{25}$$

$$\Rightarrow \quad \frac{(x-4)^2}{4} = \frac{29}{25}$$

$$1+\frac{4}{25} = \frac{25}{25}+\frac{4}{25} = \frac{29}{25}$$

$$\Rightarrow \quad \sqrt{\frac{(x-4)^2}{4}} = \pm\sqrt{\frac{29}{25}}$$

$$\Rightarrow \quad \frac{x-4}{2} = \pm\frac{\sqrt{29}}{5}$$

$$\Rightarrow \quad x-4 = \pm\frac{2\sqrt{29}}{5}$$

$$\Rightarrow \quad x = 4 \pm \frac{2\sqrt{29}}{5}$$

x-intercept(s): $\left(4+\frac{2\sqrt{29}}{5}\right), \left(4-\frac{2\sqrt{29}}{5}\right)$

or $\approx$ (6.2,0), (1.8,0)

To find the y-intercepts, plug in 0 for x and solve:

$$\frac{(0-4)^2}{4} - \frac{(y+2)^2}{25} = 1$$

$$\Rightarrow \quad \frac{16}{4} - \frac{(y+2)^2}{25} = 1$$

$$\Rightarrow \quad 4 - \frac{(y+2)^2}{25} = 1$$

$$-4 \qquad\qquad -4$$

$$\Rightarrow \quad -\frac{(y+2)^2}{25} = -3$$

$$\Rightarrow \quad \frac{(y+2)^2}{25} = 3$$

$$\Rightarrow \qquad \sqrt{\frac{(y+2)^2}{25}} = \pm\sqrt{3}$$

$$\Rightarrow \qquad \frac{y+2}{5} = \pm\sqrt{3}$$

$$\Rightarrow \qquad y + 2 = \pm 5\sqrt{3}$$

$$\Rightarrow \qquad y = -2 \pm 5\sqrt{3}$$

y-intercept(s): $\left(0, -2 + 5\sqrt{3}\right), \left(0, -2 - 5\sqrt{3}\right)$

or $\approx$ $(0, 6.7), (0, -10.7)$

Now we will turn our attention to recognizing conics. In order to know the technique that applies when graphing, we will need to recognize the shape from the form of the equation. Recognition of forms is an important skill for graphing, but this skill is also important in calculus for evaluating integrals. It is a skill that allows us to classify information quickly and access the associated techniques files away in our brains!

<u>Recognizing Conics</u>

The form of each type of conic is slightly different and these differences will help you to recognize the type of conic that you have. For example, the parabola has a square on one variable, but not the other. The other three conics have square on both variables. To distinguish between them, look at the sign between those squares. If it is addition, then you have an ellipse and if it subtraction, then you have a hyperbola. If it is addition and $a = b$, then you have a special ellipse called a circle. Sometimes you may need to complete the square(s) before you can make a decision.

<u>Examples</u>

For each of the following equations, identify the conic that it represents.

1. $y = (x-2)^2 + 3$

   This is a **parabola** because there is a square on one variable but not the other.

2. $(x-2)^2 - (y-3)^2 = 9$

   This one is a **hyperbola** because we see subtraction between the squares. We can put it in graphing form to verify:

$$\frac{(x-2)^2 - (y-3)^2}{9} = \frac{9}{9}$$

$$\Rightarrow \quad \frac{(x-2)^2}{9} - \frac{(y-3)^2}{9} = 1$$

3. $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{9} = 1$

   This one is a **circle** since there is addition between the squares and $a = b = 3$.

4. $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

   This one is an **ellipse** since there is addition between the squares and $a \neq b$ (since $a = 2$ and $b = 3$).

5. $x + 9 = 3y^2$

This is a **parabola** because there is a square on one variable but not the other.

6. $x^2 + y^2 - 6x + 4y - 30 = 0$

This one looks like an ellipse or a circle due to the addition between squares, but we need to complete the square to be sure and to identify which conic it is:

$$x^2 + y^2 - 6x + 4y - 30 = 0$$

⇨ $$x^2 - 6x + y^2 + 4y = 30$$

⇨ $$x^2 - 6x + \mathbf{9} + y^2 + 4y + \mathbf{4} = 30 + \mathbf{9} + \mathbf{4}$$

⇨ $$(x-3)^2 + (y+2)^2 = 43$$

This is clearly a **circle**.

Note: If we divide both sides by 43, we would see it as the ellipse with $a = b = \sqrt{43} \approx 6.6$

$$\frac{(x-3)^2}{43} + \frac{(y+2)^2}{43} = 1$$

7. $4x^2 - 9y^2 - 72 = 0$

We can already tell that this is a **hyperbola** because there is subtraction between the square terms, but if we wanted to put it in graphing form, we would have:

$$4x^2 - 9y^2 = 72$$

$$\Rightarrow \quad \frac{4x^2}{72} + \frac{9y^2}{72} = \frac{72}{72}$$

$$\Rightarrow \quad \frac{x^2}{18} + \frac{y^2}{8} = 1$$

8. $4y^2 + 20x^2 + 1 = 8y - 5x^2$

First, organize your terms for completing the squares:

$$4y^2 - 8y + 25x^2 = -1$$

We can already tell that this will likely be an ellipse, but we have to make sure it is defined by completing the square:

$$\Rightarrow \quad 4(y^2 - 2y) + 25x^2 = -1$$

To complete the square, we must pull out the coefficient of the square term. When you add a number inside of the parentheses, don't forget to multiply that number by the coefficient outside before adding it to the other side. Otherwise, you will not be adding the same thing to both sides.

$$\Rightarrow \quad 4(y^2 - 2y + \mathbf{1}) + 25x^2 = -1 + \mathbf{4}$$

$$\Rightarrow \quad 25x^2 + 4(y-1)^2 = 3$$

If we had ended up with a negative number on this side, we would not have had a graph at all.

$$\Rightarrow \quad \frac{25x^2 + 4(y-1)^2}{3} = \frac{3}{3}$$

$$\Rightarrow \qquad \frac{25(x-1)^2}{3} + \frac{4(y+2)^2}{3} = 1$$

$$\Rightarrow \qquad \frac{(x-1)^2}{\frac{3}{25}} + \frac{(y+2)^2}{\frac{3}{4}} = 1$$

or
$$\frac{(x-1)^2}{0.12} + \frac{(y+2)^2}{0.75} = 1$$

Now, we can see that this is an **ellipse** centered at $(1, -2)$ with $a = \sqrt{0.12} \approx 0.35$ and $b = \sqrt{0.75} \approx 0.87$.