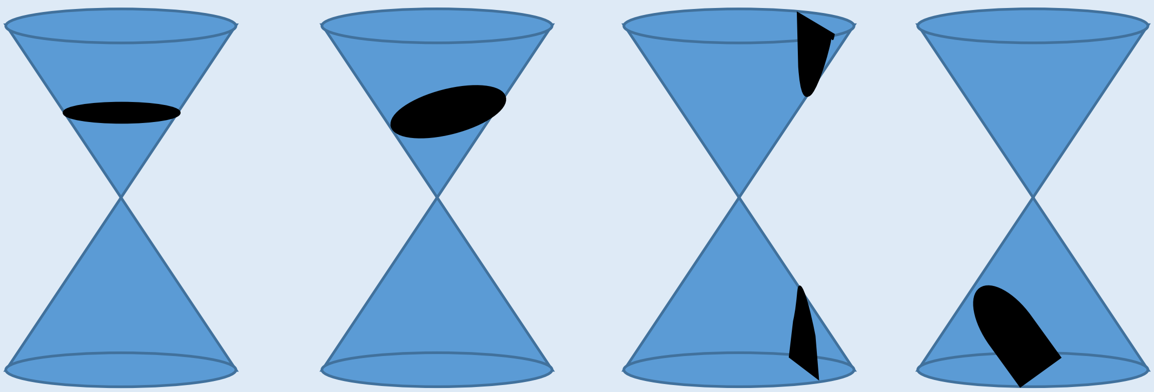
6.2 Graphing Circles and Ellipses

There are four types of *conic sections* and we have already explored one of them – parabolas. The other three are circles, ellipses and hyperbolas. In this section, we will study circles and ellipses since they have similar forms and in the next section we will study hyperbolas.

If you have never heard of a *conic section*, imagine a double cone and then imagine making a straight "cut" through it. The cross section, or slice, that you obtain from any such "cut" is called a conic section.



If you make the cut horizontally, you will get a circle. (The equation of a circle centered at the origin looks like $x^2 + y^2 = r^2$ where r is the radius, or the distance from the center to the edge.)

If you make a slightly slanted cut, you will get an ellipse. (The equation of an ellipse centered at the origin looks like $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are the distances from the center in the horizontal and vertical directions.)

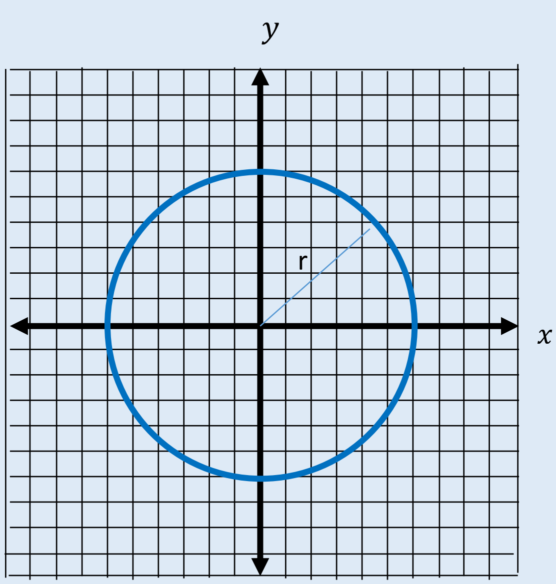
If you make the cut vertically, you will get a hyperbola, which looks a lot like two parabolas. (The equation of a hyperbola centered at the origin looks like $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where a and b are the distances from the center in the horizontal and vertical directions.)

If you make the cut with a larger slant you will eventually touch the end of the cone and this will form a parabola. The equation of a parabola with vertex at the origin looks either like $y = x^2$ or $x = y^2$

Conic sections occur in many fields and applications. Circles are important in trigonometry and engineering. Ellipses tend to show up anytime an orbit is involved (e.g. orbits of planets around a star, orbits of electrons around a nucleus, etc.) and in the reflection of light and sound waves. Hyperbolas are present in the study of radio signals and shock waves. Parabolas describe projectile motion, and are essential for optics, satellite transmission, and telescope design. Without conic sections, we would be missing much of the technology we take for granted today, including GPS (global positioning systems) and our ability to study the universe!

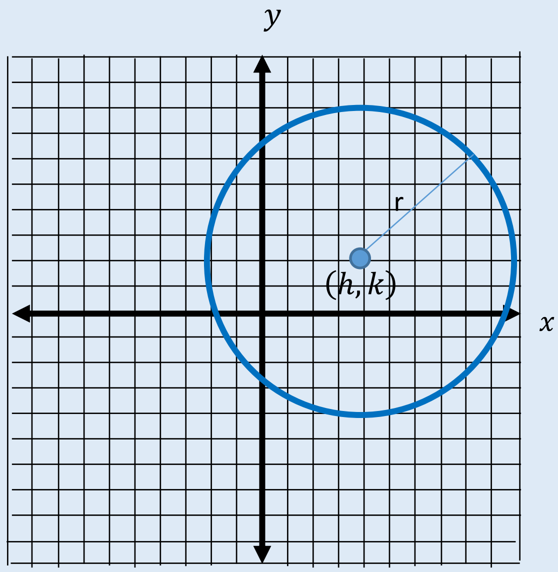
In the pictures above, you see the entire cross section, but we will actually be graphing the boundaries of these in two dimensions, so they won't be filled in. We already did this for the parabolas. We also shifted the parabolas based on the vertex (h, k) and we will do this for the other conics as well, but you will see the equations are still of the same form as indicated above, just slightly altered by the shifts.

Circles



The circle centered at the origin $(0,0)$ with radius r is pictured to the left. The *radius* is the distance from the center to the circle. It is the same for every point on the circle.

The equation of this circle is $x^2 + y^2 = r^2$.



The circle centered at the point (h, k) with radius r is pictured to the left. Notice that we just shifted our circle h units to the right and k units up. (Also, note that the center is not part of the graph of the circle. It is just a place to start.)

The equation of this circle is $(x-h)^2 + (y-k)^2 = r^2$.

The general method to graph a circle is to first find the center and the radius from the equation. If your equation is already in graphing form $(x-h)^2 + (y-k)^2 = r^2$, then the center and the radius can be easily be found by inspection. (If it is not in this form, you may need to put it in this form by completing the square on either one or both of the variables.)

Once you have the center and the radius, just start at the center and move out a distance of r in each direction (up, down, left and right) to get 4 points on the circle. Even if your radius is not a whole number, you can approximate using a decimal (e.g. if $r = \sqrt{7}$, then round it to $r = 2.6$) for graphing purposes.

Finally, trace the circle through these four points. It is really that easy!

Examples

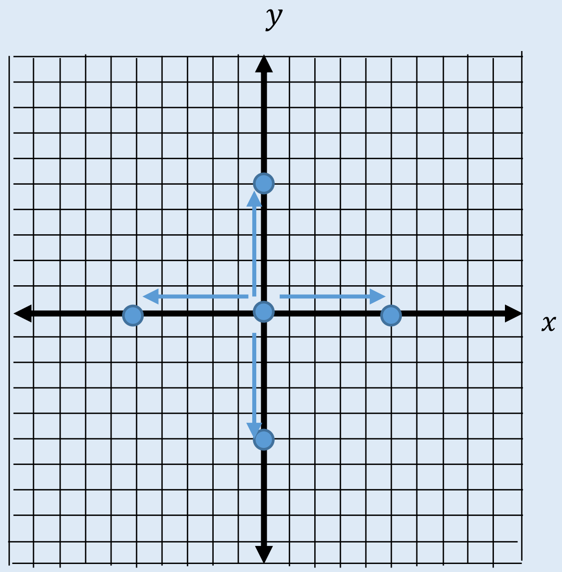
Graph each of the following relations and give the domain, range, x-intercept(s) and y-intercept(s).

1. $x^2 + y^2 = 25$

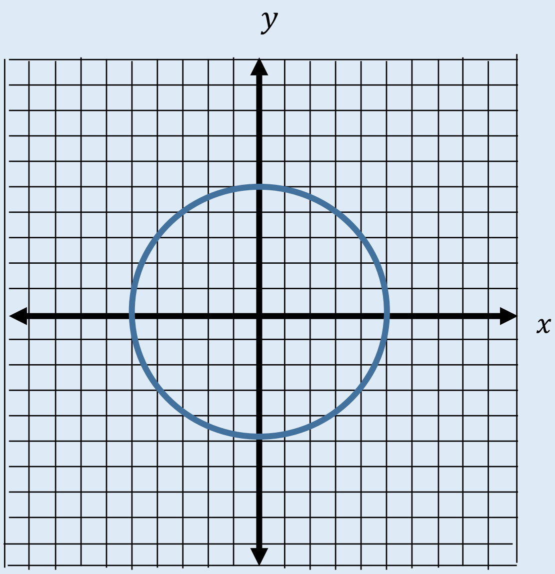
This circle is already in graphing form with center $(0,0)$ and $r = 5$. Note that the equation could be written in the following form:

$$(x-0)^2 + (y-0)^2 = 5^2$$

We will begin at the center $(0,0)$ and move a distance of 5 in each direction to get 4 points on the circle:



Now trace out the circle by connecting the four outer points. The center is not part of the circle, so erase that point.



Now, we can give the requested information:

domain: $[-5,5]$

range: $[-5,5]$

x-intercept(s): $(5,0), (-5,0)$

y-intercept(s): $(0,5), (0,-5)$

In this case, the information was very easy to find from the graph, but in general, you can add and subtract the value of r to the coordinates of the center to get the domain and the range, respectively. The domain of a circle will always be $[h-r, h+r]$ since those values give us the horizontal boundaries of the circle. Similarly, the range of a circle will always be $[k-r, k+r]$ since those values give us the vertical boundaries of the circle. The intercepts can be found in the usual way by setting the *other* variable equal to 0 in the equation that describes the relation. When the circle is centered at the origin, all of this is obvious from the graph, though.

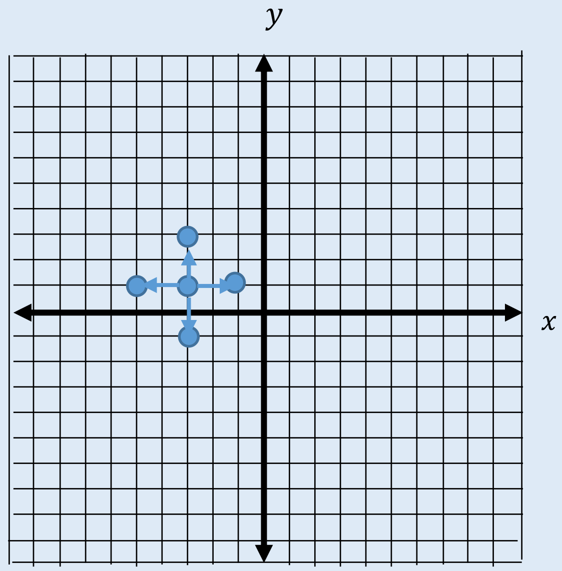
2. $(x+3)^2 + (y-1)^2 = 4$

This circle is already in graphing form with center $(-3,1)$ and $r = 2$. (We take the opposite signs of the values inside of the parentheses to get the center.)

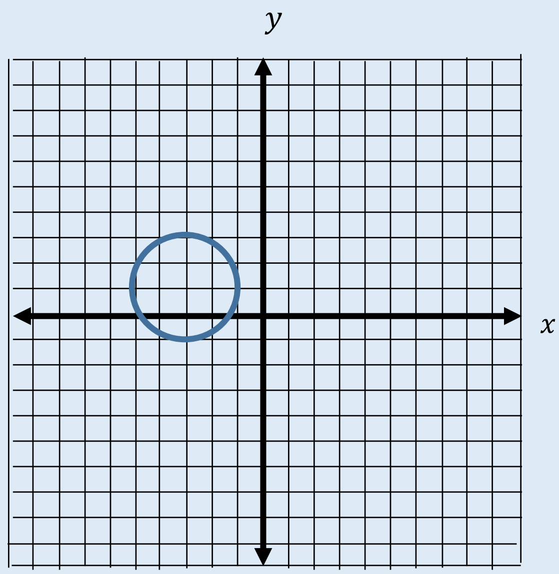
Note that the equation could be written in the following form:

$$\left(x - (-3)\right)^2 + (y-1)^2 = 2^2$$

We will begin at the center $(-3,1)$ and move a distance of 2 in each direction to get 4 points on the circle:



Now trace out the circle by connecting the four outer points. The center is not part of the circle, so erase that point.



We can give some of the requested information from the graph:

domain: $[-5,-1]$

range: $[-1,3]$

y-intercept(s): none

For domain: $h - r = -3 - 2 = -5$ *and* $h + r = -3 + 2 = -1$

For range: $k - r = 1 - 2 = -1$ *and* $k + r = 1 + 2 = 3$

We can see from the graph that there are no y-intercepts, but we need to solve an equation to get the x-intercepts. Plugging 0 in for y, we get:

$$(x+3)^2 + (0-1)^2 = 4$$

⇨ $$(x+3)^2 + 1 = 4$$

⇨ $$(x+3)^2 = 3$$

⇨ $$x + 3 = \pm\sqrt{3}$$

⇨ $$x = -3 \pm \sqrt{3}$$

x-intercept(s): $\left(-3+\sqrt{3}, 0\right), \left(-3-\sqrt{3}, 0\right)$

or $\approx$ $(-1.3,0), (-4.7,0)$

3. $x^2 + y^2 + 8x - 6y = 11$

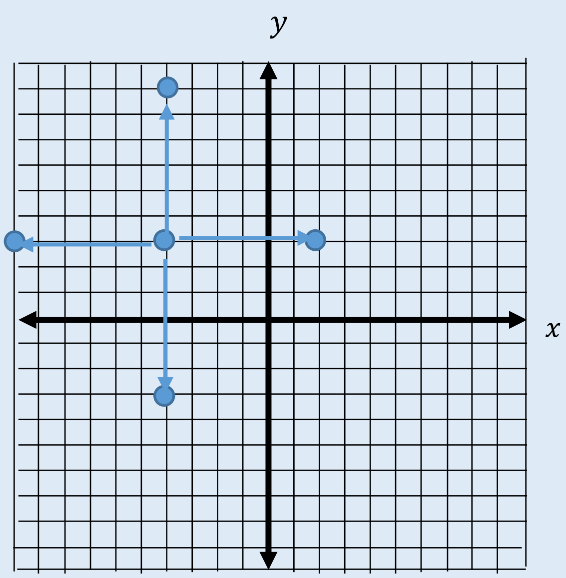
This example is not in standard graphing form, so we need to complete the square on both x and y before we can find the center and the radius. It is convenient to add to both sides rather than add and subtract to one side here, so we will approach it that way.

$$x^2 + 8x + y^2 - 6y = 11$$

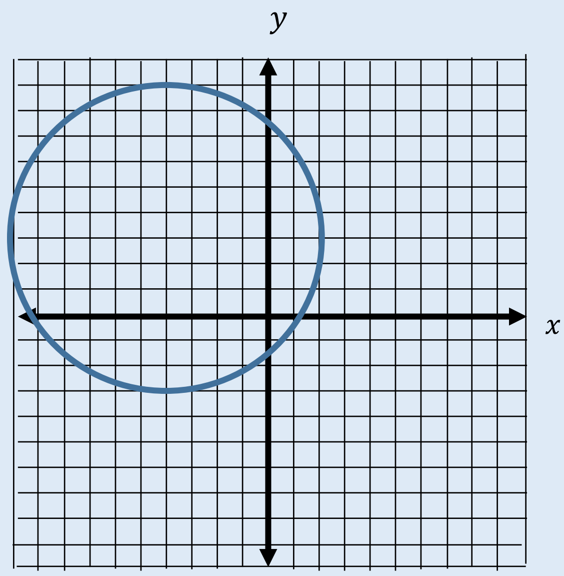
⇨ $$x^2 + 8x + \mathbf{16} + y^2 - 6y + \mathbf{9} = 11 + \mathbf{16} + \mathbf{9}$$

⇨ $$(x + 4)^2 + (y - 3)^2 = 36$$

Now we can see that this is a circle with center $(-4,3)$ and $r = 6$. We will begin at the center $(-4,3)$ and move a distance of 6 in each direction to get 4 points on the circle:



Now trace out the circle by connecting the four outer points. The center is not part of the circle, so erase that point.



Now, we can give the requested information:

domain: $[-10,2]$

range: $[-3,9]$

For domain: $h - r = -4 - 6 = -10$ *and* $h + r = -4 + 6 = 2$

For range: $k - r = 3 - 6 = -3$ *and* $k + r = 3 + 6 = 9$

We can see from the graph that there are two x-intercepts and two y-intercepts. To find the x-intercepts, plug in 0 for y in either form of the equation. We find it easier to plug into the completed square form here:

$$(x+4)^2 + (0-3)^2 = 36$$

⇨ $$(x+4)^2 + 9 = 36$$

⇨ $$(x+4)^2 = 27$$

⇨ $$x + 4 = \pm\sqrt{27} = \pm 3\sqrt{3}$$

⇨ $$x = -4 \pm 3\sqrt{3}$$

x-intercept(s): $\left(-4+3\sqrt{3}, 0\right), \left(-4-3\sqrt{3}, 0\right)$

or $\approx$ $(1.2,0), (-9.2,0)$

To find the y-intercepts, plug in 0 for x in either form of the equation:

$$(0+4)^2+(y-3)^2=36$$

⇨ $$16+(y-3)^2=36$$

⇨ $$(y-3)^2=20$$

⇨ $$y-3=\pm\sqrt{20}=\pm 2\sqrt{5}$$

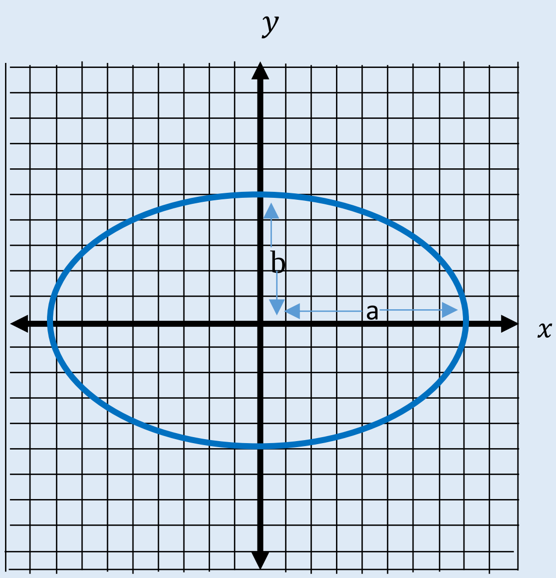
⇨ $$x=3\pm 2\sqrt{5}$$

y-intercept(s): $\left(0,3+2\sqrt{5},\right),\left(0,3-2\sqrt{5}\right)$

or $\approx$ $(0,7.5),(0,-1.5)$

Ellipses

We now turn our attention to the ellipse, which is very similar to the circle. In fact, a circle is a type of ellipse where the distance travelled in each direction from the center is the same (the radius). An ellipse can be thought of like a "squashed circle" since it is longer in one direction than the other.



An ellipse centered at the origin $(0,0)$ is pictured to the left.

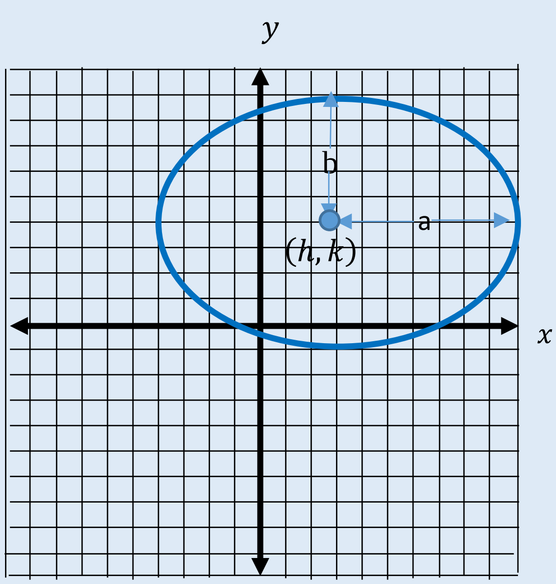
The equation of this ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The longer axis is called the *major axis* and the shorter axis is called the *minor axis*. The length of the major axis in the picture above is $2a$, while the length of the minor axis is $2b$. Some textbooks always use a for the major axis and b for the minor axis, but we prefer to use a to denote the horizontal axis (in the x direction) and b to denote the vertical axis (in the y direction) as the association seems more natural.

Notice that a circle centered at the origin could also be written in this form as follows:

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

For this "ellipse", $a = r$ and $b = r$, which means that the distance from the center in the horizontal direction is the same as the distance in the vertical direction. That means we have a special kind of ellipse, called a circle.

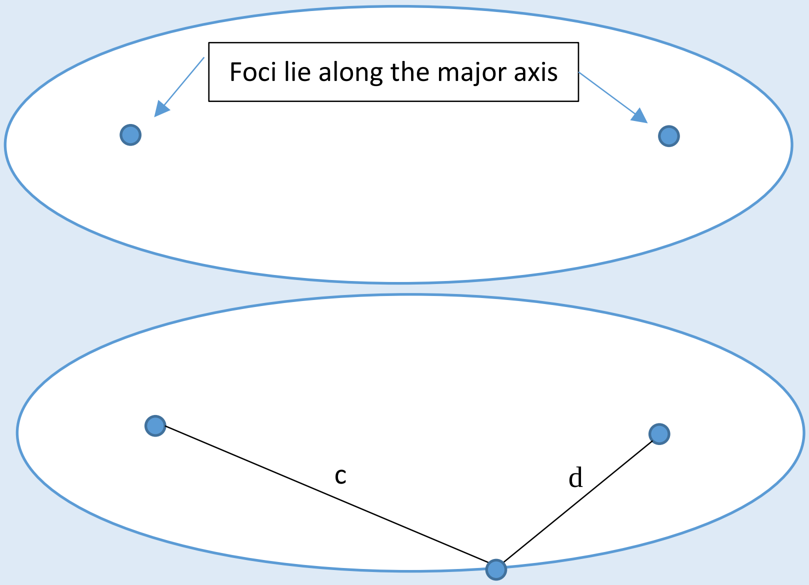


An ellipse centered at the point (h, k) is pictured to the left. Notice that this ellipse is shifted h units to the right and k units up.

The equation of this ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

There are many other interesting things that can be said about ellipses. They have two special points inside of them called *foci* (each one called a *focus*). The planets in our solar system have elliptical orbits around the sun. The sun resides at a focus for each of these ellipses.



Neat facts about foci:

1. If you sum up their distances to the same point on the ellipse, you will always get the same value. (c+d is always the same)
2. They act as the "centers" of stable orbits.
3. Any sound wave emitted from one of the foci will bounce off of the ellipse and end up at the other focus. (This is how we use ellipses in medical science to break up kidney stones, for example).

Although we will not be focusing on the foci (no pun intended) or any of the really interesting applications in this section, they are worth mentioning and you will encounter many applications in your science curriculum. If you are interested, there are many websites with great information on the applications. All you need to do is perform a search. We will keep our attention on graphing so as to not overwhelm the student with too many new ideas at once.

The only difference between graphing a circle and graphing an ellipse is the distance that you move from the center to get points on the ellipse is different in the horizontal direction than in the vertical direction whereas for the circle it is the same in all directions. You can get the distance in the horizontal direction by taking the square root of the number in the denominator of the x term and similarly, you can get the distance in the vertical direction by taking the square root of the number in the denominator of the y term. The center is read off of the equation in the usual way.

Examples

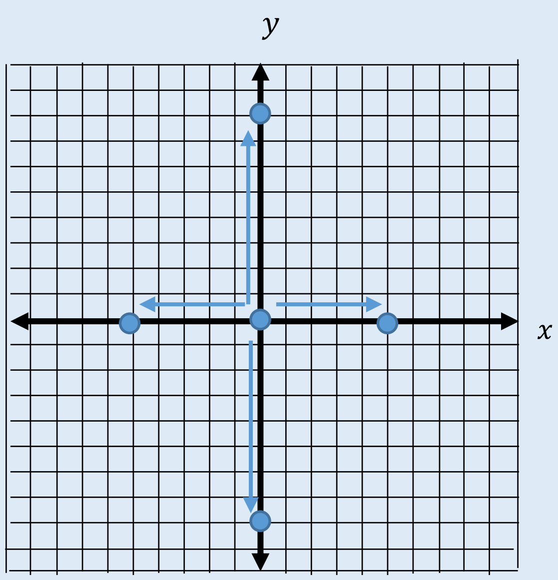
Graph each of the following relations and give the domain, range, x-intercept(s) and y-intercept(s).

4. $\frac{x^2}{25}+\frac{y^2}{64}=1$

This ellipse is already in graphing form with center $(0,0)$.
Also, $a=5$ since $\sqrt{25}=5$ and $b=8$ since $\sqrt{64}=8$.

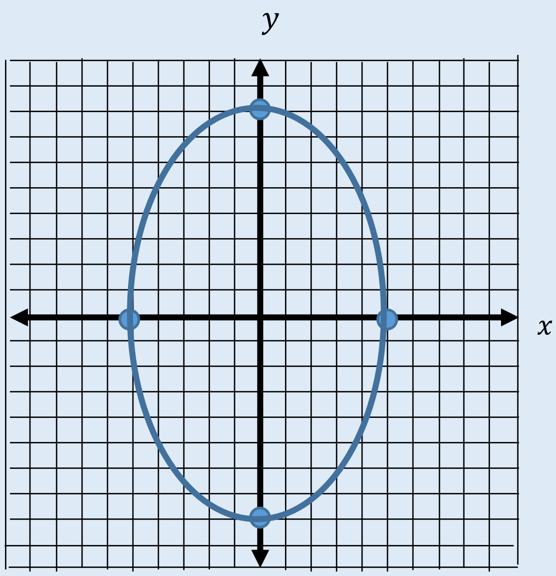
Note that the equation could be written in the following form:

$$\frac{(x-0)^2}{5^2}+\frac{(y-0)^2}{8^2}=1$$



We will begin at the center $(0,0)$ and move a distance of 5 in the horizontal direction and a distance of 8 in the vertical direction to get 4 points on the ellipse.

Now trace out the ellipse by connecting the four outer points. The center is not part of the ellipse, so erase that point.



Now, we can give the requested information:

domain: $[-5,5]$

range: $[-8,8]$

x-intercept(s): $(5,0), (-5,0)$

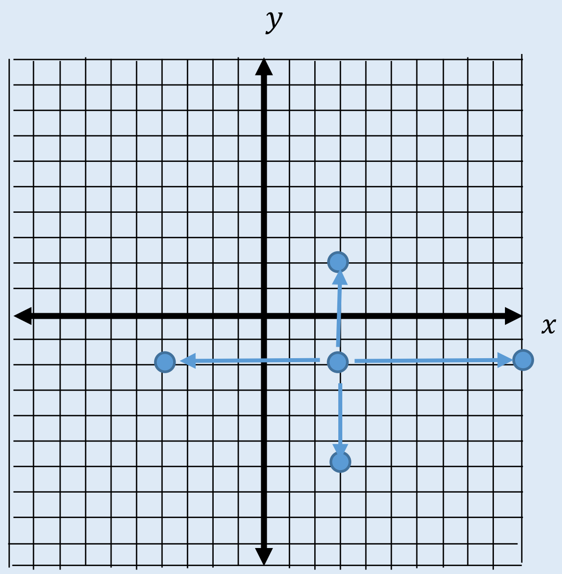
y-intercept(s): $(0,8), (0,-8)$

5. $\frac{(x-3)^2}{49} + \frac{(y+2)^2}{16} = 1$

This ellipse is already in graphing form with center $(3,-2)$. Also, $a = 7$ since $\sqrt{49} = 7$ and $b = 16$ since $\sqrt{16} = 4$.

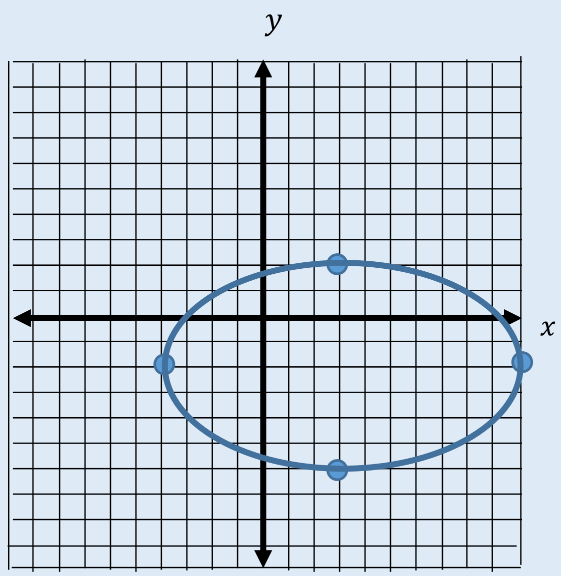
Note that the equation could be written in the following form:

$$\frac{(x-3)^2}{7^2} + \frac{(y+2)^2}{4^2} = 1$$



We will begin at the center $(3, -2)$ and move a distance of 7 in the horizontal direction and a distance of 4 in the vertical direction to get 4 points on the ellipse.

Now trace out the ellipse by connecting the four outer points. The center is not part of the ellipse, so erase that point.



Now, we can give the requested information:

domain: $[-4,10]$

range: $[-6,2]$

For domain: $h-a=3-7=-4$ *and* $h+a=3+7=10$

For range: $k-b=-2-4=-6$ *and* $k+b=-2+4=2$

We can see from the graph that there are two x-intercepts and two y-intercepts. To find the x-intercepts, plug in 0 for y:

$$\frac{(x-3)^2}{49}+\frac{(0+2)^2}{16}=1$$

$$\Rightarrow \quad \frac{(x-3)^2}{49}+\frac{4}{16}=1$$

Reducing fractions as you go makes for easier computations.

$$\Rightarrow \quad \frac{(x-3)^2}{49}+\frac{1}{4}=1$$

$$\qquad -\frac{1}{4} \quad -\frac{1}{4}$$

$$1-\frac{1}{4}=\frac{4}{4}-\frac{1}{4}=\frac{3}{4}$$

$$\Rightarrow \quad \frac{(x-3)^2}{49}=\frac{3}{4}$$

$$\Rightarrow \quad \sqrt{\frac{(x-3)^2}{49}}=\pm\sqrt{\frac{3}{4}}$$

$$\Rightarrow \quad \frac{x-3}{7}=\pm\sqrt{\frac{3}{4}}$$

$$\Rightarrow \quad \frac{x-3}{7}=\pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x-3=\pm\frac{7\sqrt{3}}{2}$$

$$\Rightarrow \quad x=3\pm\frac{7\sqrt{3}}{2}$$

x-intercept(s): $\left(3+\frac{7\sqrt{3}}{2},0\right),\left(3-\frac{7\sqrt{3}}{2},0\right)$

or $\approx$ $(9.1,0),(-3.1,0)$

To find the y-intercepts, plug in 0 for x in either form of the equation:

$$\frac{(0-3)^2}{49}+\frac{(y+2)^2}{16}=1$$

$$\Rightarrow \quad \frac{9}{49}+\frac{(y+2)^2}{16}=1$$

$$\Rightarrow \quad \frac{9}{49}+\frac{(y+2)^2}{16}=1$$

$$\underline{-\frac{9}{49} \qquad\qquad -\frac{9}{49}}$$

$$1-\frac{9}{49}$$

$$=\frac{49}{49}-\frac{9}{49}$$

$$=\frac{40}{49}$$

$$\Rightarrow \quad \frac{(y+2)^2}{16}=\frac{40}{49}$$

$$\Rightarrow \quad \sqrt{\frac{(y+2)^2}{16}}=\pm\sqrt{\frac{40}{49}}$$

$$\Rightarrow \quad \frac{y+2}{4}=\pm\sqrt{\frac{40}{49}}$$

$$\Rightarrow \quad \frac{y+2}{4}=\pm\frac{2\sqrt{10}}{7}$$

$$4\cdot\frac{2\sqrt{10}}{7}$$

$$=\frac{4}{1}\cdot\frac{2\sqrt{10}}{7}$$

$$=\frac{8\sqrt{10}}{7}$$

$$\Rightarrow \quad y+2=\pm\frac{8\sqrt{10}}{7}$$

$$\Rightarrow \quad y=-2\pm\frac{8\sqrt{10}}{7}$$

y-intercept(s): $\left(0,-2+\frac{8\sqrt{10}}{7}\right),\left(0,-2-\frac{8\sqrt{10}}{7}\right)$

or $\approx$ $(0,1.6),(0,-5.6)$

For some of the ellipse examples, you will need to put them in graphing form before you can find the center along with a and b. Sometimes you will need to complete the square and other times you will need to just divide through by a number to get 1 on the other side. In the next two examples, we will illustrate this once more, but we leave it to the reader to graph these and find the information for them.

6. $4(x+6)^2 + 25(y+3)^2 = 100$

For this one, the squares are already completed, but it is still not in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. We need to get 1 on the right side of the equation, so divide both sides by whatever is there and then simplify:

$$\frac{4(x+6)^2 + 25(y+3)^2}{100} = \frac{100}{100}$$

$$\Rightarrow \quad \frac{4(x+6)^2}{100} + \frac{25(y+3)^2}{100} = 1$$

$$\Rightarrow \quad \frac{(x+6)^2}{25} + \frac{(y+3)^2}{4} = 1$$

Now, it is in the correct form for reading off the information we need to graph the ellipse. The center is $(-6, -3)$, $a = 5$ and $b = 2$.

7. $9x^2 + 4y^2 - 18x + 16y = 11$

This one is not even recognizable as an ellipse yet, but we can complete the square on both x and y to see that it takes the form of an ellipse equation:

$$9x^2 - 18x + 4y^2 + 16y = 11$$

Rearrange your terms, grouping x's and y's together. Any numbers should be moved to the other side.

$$\Rightarrow \quad 9(x^2 - 2x) + 4(y^2 + 4y) = 11$$

To complete the square, we must pull out the coefficient of the square term. When you add a number inside of the parentheses, don't forget to multiply that number by the coefficient outside before adding it to the other side. Otherwise, you will not be adding the same thing to both sides.

$$\Rightarrow \quad 9(x^2 - 2x + \mathbf{1}) + 4(y^2 + 4y + \mathbf{4}) = 11 + \mathbf{9} + \mathbf{16}$$

$$\Rightarrow \quad 9(x-1)^2 + 4(y+2)^2 = 36$$

After completing the square, get 1 on the right hand side by dividing both sides by 36. Then simplify.

$$\Rightarrow \quad \frac{9(x-1)^2 + 4(y+2)^2}{36} = \frac{36}{36}$$

$$\Rightarrow \quad \frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = \frac{36}{36}$$

$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

Now, it is in the correct form for reading off the information we need to graph the ellipse. The center is $(1, -2)$ and $a = 2$ and $b = 3$.