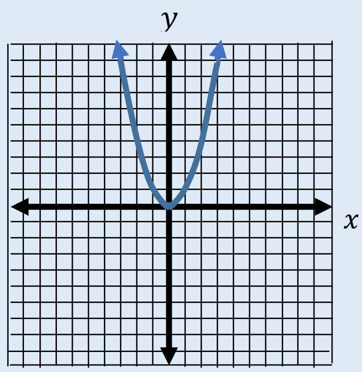
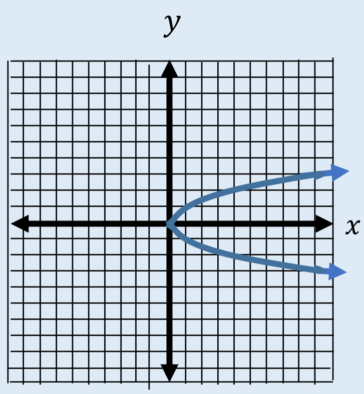
6.1 Graphing Parabolas

We already know how to graph basic parabolas which open up or down and which can be shifted around. Now, we will generalize what we know in order to include parabolas which open sideways (left or right) and we will also be able to graph wide and narrow parabolas. We will utilize the process of completing the square in order to put our quadratics into graphing form, so you may want to review section 2.8 as well. We begin by comparing our basic parabola $y = x^2$ with the basic sideways parabola $x = y^2$. Notice that x and y are switched in these two equations. (This is not a coincidence and it is a property of *inverse relations*, which we study in more detail in section 6.5.)

$y = x^2$



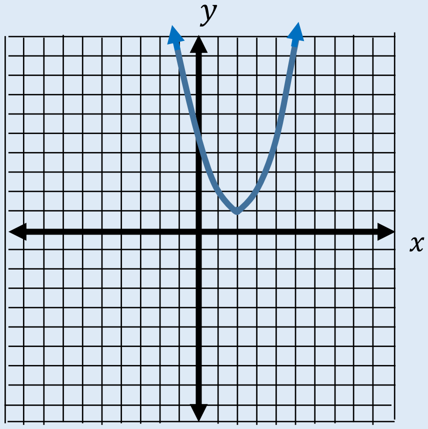
$x = y^2$



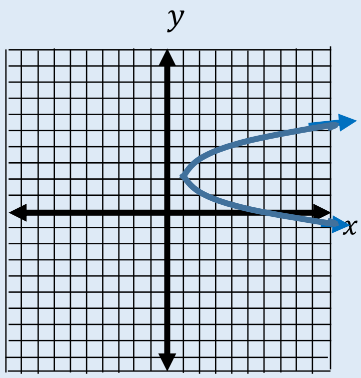
Notice that the graph of $x = y^2$ could also be thought of as the two graphs we get by solving this equation for y: $x = y^2$ implies $y = \pm\sqrt{x}$, with $y = \sqrt{x}$ representing the top half of the graph and $y = -\sqrt{x}$ representing the bottom half.

In the same way that we were able to shift the graph of $y = x^2$ around, we can also perform the same kinds of transformations on $x = y^2$.

Consider the graph of $y = (x - 2)^2 + 1$, for instance. We know that one way of obtaining this graph is by first graphing our basic graph, $y = x^2$, and then shifting it to the right by 2 and up 1 to obtain the following graph:



Similarly, we could consider the graph of $x = (y - 2)^2 + 1$, but to shift this, we need to keep in mind which shift is attached to which variable. For example, here the -2 belongs to y, and since it is on the same side as y, we end up going in the opposite direction (because if you solve for y, you will end up adding 2 to the other side). Hence the graph will shift *up* 2. The value of 1 belongs to x here, and we will end up shifting in the *same* direction since the equation is already solved for x. So the graph will shift to the right by 1:



One way to remember this process that matches what you already know is to move the opposite direction for what is in the parentheses and the same direction for what is outside of the parentheses.

We will not need to spend much time on this because there is a more general way to graph parabolas that will include the parabolas that are wider and narrower that these two basic parabolas. In order to graph ALL parabolas, we need to allow for more than shifting and reflecting because there is also "stretching or shrinking" involved for the ones that have a coefficient other than 1 for the square term. We can take care of this "stretching or shrinking" by plotting a couple of points (one of these has to be the *vertex*) and using symmetry to finish our graph.

If we are going to use this method, we must have the parabola in graphing form and we can put the parabola in graphing form by completing the square.

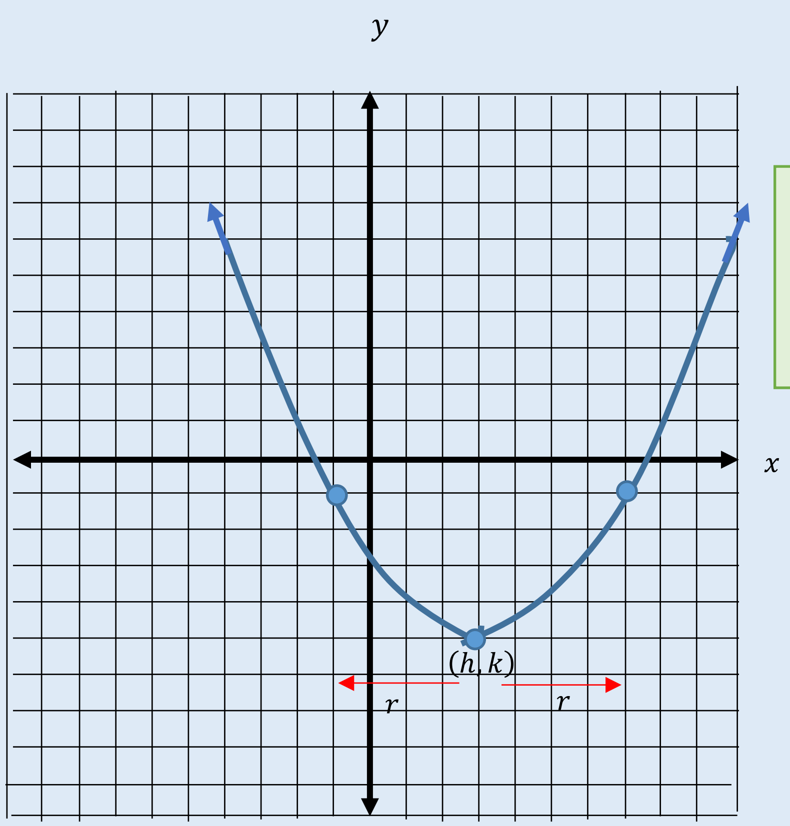
For parabolas that open up or down, standard graphing form is as follows**:** $\boldsymbol{y = a(x - h)^2 + k}$

From this form, we can read off the ***vertex*** $\boldsymbol{(h, k)}$ which is going to be the highest or lowest point on the parabola – the place at which it "turns around". (Note that to get the vertex, you take the opposite sign of what is in the parentheses and the same sign of what is outside the parentheses, just as when you are shifting. It is for the same reason.)

You can also read off the ***axis of symmetry*** $\boldsymbol{x = h}$, which is the vertical line that divides the parabola into two equal halves. It is the line that acts like a mirror on one side of the parabola to give us the other side. The axis of symmetry is not part of the graph of the parabola, but sometimes it is requested in the homework problems.

You can also tell whether the graph opens up or down by looking at the sign on a. If a is positive, then the graph opens up. If a is negative, then the graph opens down. This really just helps you double check that you graphed it correctly, as plotting the points should make it open the correct way if there are no numerical errors.

Once you read off the vertex, you can choose another value to plug in for x. (Since the domain for these types of parabolas is "all real numbers", you can choose any value that you like). Plug your chosen value into the equation to get the corresponding y value and then plot your point (x, y). Then, count how far that value is away from h and if you go the same distance in the other direction, you know the y-value will be the same.



The y values should match for $h + r$ and $h - r$, where r is the distance you chose to go from h to plug in another value.

The best way to make this process clear is with some examples, so first we will consider examples of parabolas that open up or down. Within the examples, we will also review the concepts of domain, range, and intercepts as we usually do.

<u>Examples</u>
For each of the following, graph the given relation. Give the vertex, axis of symmetry, domain, range, as well as the x and y intercept(s).

1. $y = 3(x-1)^2 - 2$

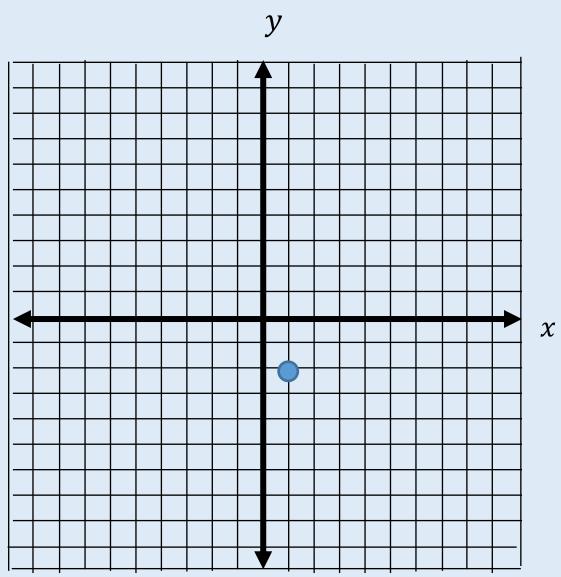
First, you will notice that this example is already in standard graphing form, so the square has already been completed. Recall that standard graphing form is $\boldsymbol{y = a(x-h)^2 + k}$.
We should be able to read off the vertex (h, k) as well as the axis of symmetry $x = h$ pretty quickly from looking at the equation:

vertex: $(1, -2)$

axis of symmetry: $x = 1$

Note again that we took the opposite sign for what was inside of the parentheses and the same sign for what was outside.

We can start graphing by plotting the vertex:

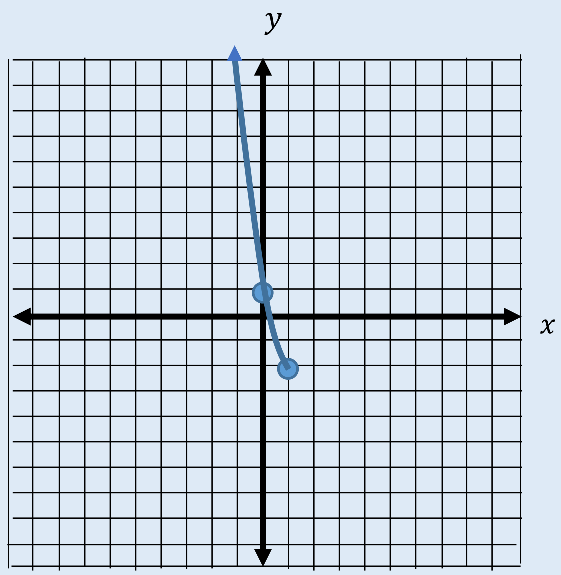


Now decide what you want to plug in next. We will choose to plug in 0 for x (since that will also give us the y-intercept at the same time).

Plugging in 0 for x, we get:

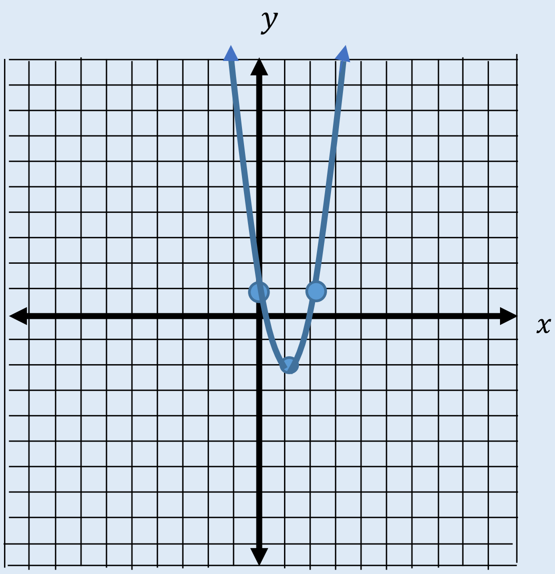
$$y = 3(0-1)^2 - 2$$
$$= 3(-1)^2 - 2$$
$$= 3 \cdot 1 - 2$$
$$= 3 - 2$$
$$= 1$$

This means we have the point $(0,1)$ on the graph as well:

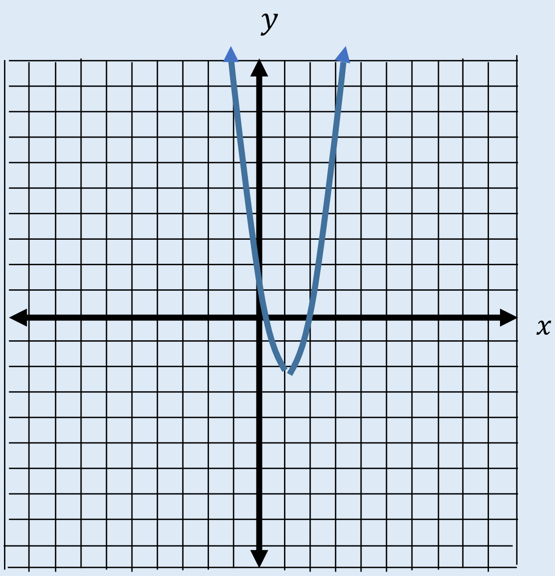


We can sketch half of the parabola using the vertex and our new point, although plugging in one more value on this side will give you a more accurate sketch. (e.g. if you plug in -1, you will get 10)

Now to get the other side of the parabola, you can use symmetry. Since you traveled a distance of 1 from the vertex to get to 0, you can go a distance of 1 in the other direction to land at 2 and you should get the same y-value. That means the point $(2,1)$ will also be on the graph. Now, we can use this point to guide us when we graph the other half of the parabola:



So our final graph (without the exaggerated points on it) looks like this:



There is a little shortcut that some teachers like to show their students that allows them to think of the value $'a'$ as a kind of “slope”, but be careful not to confuse this with graphing a line, because they are different shapes!

With that in mind, the trick is to write $'a'$ as a fraction and move up and over (in both directions) from the vertex to get two symmetric points on the parabola. In our case, we would move up 3 and over 1 in each direction to arrive at the points we happened to get by plugging in a value and using symmetry.

We can see the domain and range pretty clearly from the graph, but we also know the domain is all real numbers because this a polynomial equation in the variable x (we can plug in any real number for a polynomial variable and get out a real number). Following the graph from bottom to top, we can see that the range is $[-2, \infty)$ since the graph covers all of these y values. We already have the y-intercept also since we plugged in 0 earlier and found the point $(0,1)$.

The only thing we don't have yet are the x-intercept(s), and from the graph we see that we should have two of them. To find them, we just plug in 0 for y and solve for x, as usual:

$$0 = 3(x-1)^2 - 2$$

⇨ $$2 = 3(x-1)^2$$

⇨ $$\frac{2}{3} = (x-1)^2$$

⇨ $$\pm\sqrt{\frac{2}{3}} = x - 1$$

⇨ $$1\pm\sqrt{\frac{2}{3}} = x$$

Now rationalize your answers:

⇨ $$x = 1 \pm \frac{\sqrt{6}}{3}$$

or rounding them as decimals:

⇨ $$x = 1.816, x = 0.184$$

Now, let's write down the requested information:

vertex: $(1, -2)$

axis of symmetry: $x = 1$

domain: $(-\infty, \infty)$

range: $[-2, \infty)$

x-intercept(s): $\left(1 + \frac{\sqrt{6}}{3}, 0\right), \left(1 - \frac{\sqrt{6}}{3}, 0\right)$

y-intercept(s): $(0,1)$

For the rest of the examples, we will give the requested information without as much detail since that part of the problem is review from prior sections.

2. $y = -3(x+2)^2 + 4$

Again, you will notice that this example is already in standard graphing form, so we should be able to read off the vertex (h, k) as well as the axis of symmetry $x = h$ pretty quickly from looking at the equation:

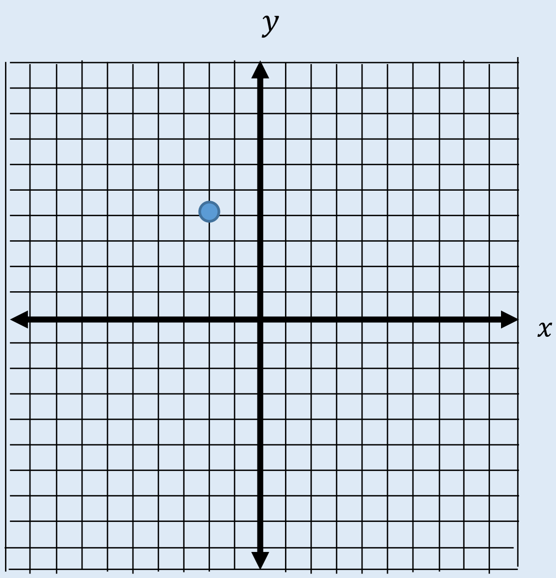
vertex: $(-2,4)$

axis of symmetry: $x = -2$

Note again that we took the opposite sign for what was inside of the parentheses and the same sign for what was outside.

We also note that this one will open down since $a = -3$.

We can start graphing by plotting the vertex:

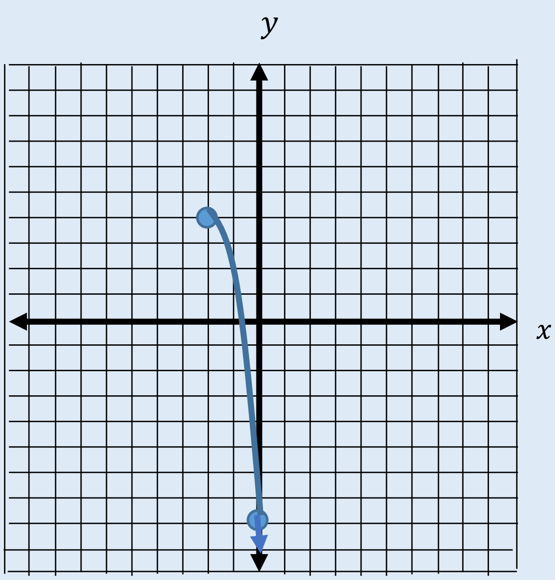


Now decide what you want to plug in next. We will choose to plug in 0 for x (since that will also give us the y-intercept at the same time).

Plugging in 0 for x, we get:

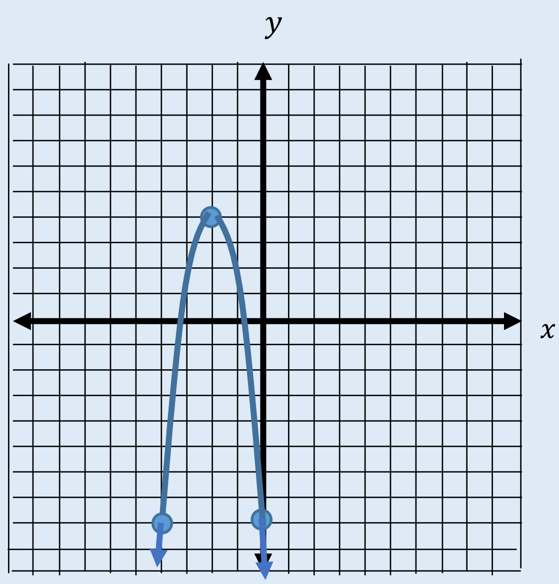
$$y = -3(0+2)^2 + 4$$
$$= -3(2)^2 + 4$$
$$= -3 \cdot 4 + 4$$
$$= -12 + 4$$
$$= -8$$

This means we have the point $(0, -8)$ on the graph as well:

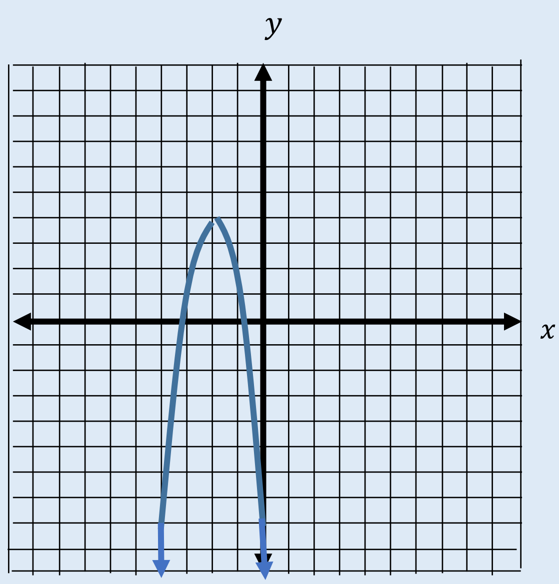


Notice that when we plug in our points, we see that it opens down as well.

Now to get the other side of the parabola, use symmetry. Since you traveled a distance of 2 from the vertex to get to 0, you can go a distance of 2 in the other direction to land at -4 and you should get the same y-value. That means the point $(-4, -8)$ will also be on the graph. Now, we can use this point to guide us when we graph the other half of the parabola:



So our final graph (without the exaggerated points on it) looks like this:



We could also have used our value of $a = \frac{-3}{1}$ to move down 3 and over 1 in each direction to arrive at two points on the parabola.

To find the x-intercept(s), plug in 0 for y and solve for x, as usual:

$$0 = -3(x+2)^2 + 4$$

⇨ $$-4 = -3(x+2)^2$$

⇨ $$\frac{4}{3} = (x+2)^2$$

⇨ $\pm\sqrt{\frac{4}{3}} = x + 2$

⇨ $-2 \pm \sqrt{\frac{4}{3}} = x$

Now rationalize your answers:

⇨ $x = -2 \pm \frac{2\sqrt{3}}{3}$

or rounding them as decimals:

⇨ $x = -0.845, x = -3.155$

Now, let's write down the requested information:

vertex: $(-2,4)$

axis of symmetry: $x = -2$

domain: $(-\infty, \infty)$

range: $(-\infty, 4]$

x-intercept(s): $\left(-2 + \frac{2\sqrt{3}}{3}, 0\right), \left(-2 - \frac{2\sqrt{3}}{3}, 0\right)$

y-intercept(s): $(0, -8)$

3. $y = -2x^2 + 8x - 14$

This example is not in graphing form, so we first need to complete the square to put it in graphing form:

⇨ $y = -2(x^2 - 4x + 7)$

⇨ $y = -2(x^2 - 4x + \mathbf{4} - \mathbf{4} + 7)$

⇨ $y = -2[(x-2)^2 + 3]$

⇨ $y = -2(x-2)^2 - 6$

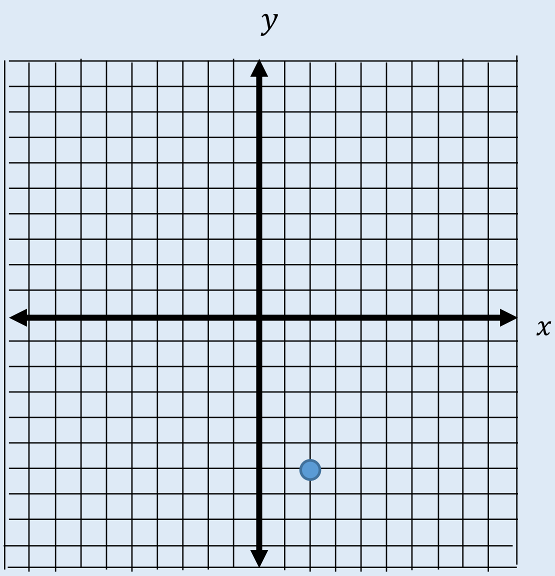
See section 2.8 for a review of this process.

vertex: $(2, -6)$

axis of symmetry: $x = 2$

We also note that this one will open down since $a = -2$.

We can start graphing by plotting the vertex:



Now decide what you want to plug in next. We will choose to plug in 0 for x (since that will also give us the y-intercept at the same time).

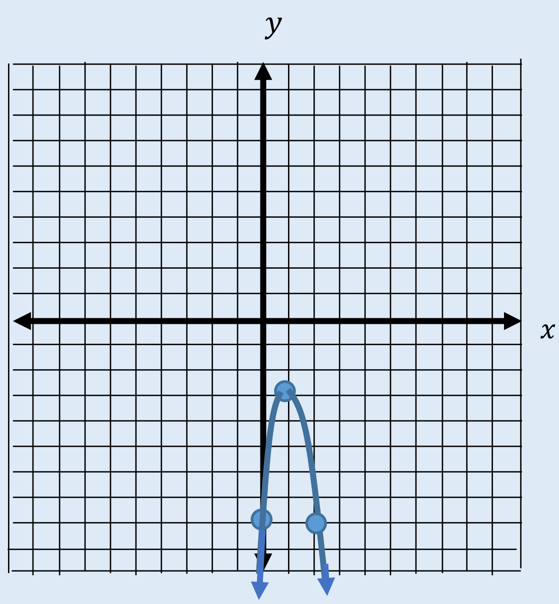
Plugging in 0 for x in the original equation gives us:

$$y = -2(0)^2 + 8 \cdot 0 - 14$$
$$= -14$$

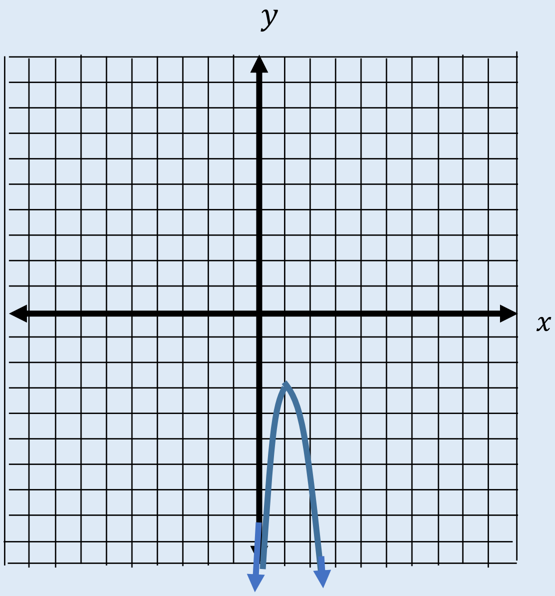
Note that we can plug values into either form of the equation, but the original form is easy to plug 0 into since the other terms get wiped out.

This means we have the point $(0, -14)$ on the graph as well. Now to get the other side of the parabola, use symmetry. Since you traveled a distance of 2 from the vertex to get to 0, you can go a distance of 2 in the other direction to land at 4 and you should get the same y-value. That means the point $(4, -14)$ will also be on the graph.

The following graph is scaled by 2's, so every box is worth 2 in each direction:



So our final graph (without the exaggerated points on it) looks like this:



We could also have used our value of $a = \frac{-2}{1}$ to move down 2 and over 1 in each direction to arrive at two points on the parabola.

Notice that there are no x-intercepts on the graph and if we try to solve the equation with y equal to 0, we will get imaginary numbers. (Our algebra should match our picture).

Now, let's write down the requested information:

vertex: $(2, -6)$

axis of symmetry: $x = 2$

domain: $(-\infty, \infty)$

range: $(-\infty, -6]$

x-intercept(s): none

y-intercept(s): $(0, -14)$

Sideways Parabolas

Now, we will turn our attention to the sideways parabolas. When you read off the vertex, you need to be careful because the value for k is inside the parentheses (associated with y) and the value for h is outside of the parentheses. We can still use symmetry to graph, but now we will be plugging in values for y and getting out values for x.

Standard graphing form for a sideways parabola is as follows:

$$\boldsymbol{x = a(y - k)^2 + h}$$

The **vertex** is still $(\boldsymbol{h, k})$**,** but notice where h and k are. You will now take the same sign for h and the opposite sign for k.

The **axis of symmetry** is the horizontal line $\boldsymbol{y = k}$.

If the value of a is positive, then the parabola will open to the right and if the value of a is negative, the parabola will open to the left.

Sometimes we have to complete the square to put our parabolas in standard graphing form, just as we do with the ones that open down. The process is the same for graphing.

4. $x = y^2 + 6y + 8$

This example is not in graphing form, so we first need to complete the square to put it in graphing form:

⇨ $x = y^2 + 6y + \mathbf{9} - \mathbf{9} + 8)$

⇨ $x = (y + 3)^2 - 1$

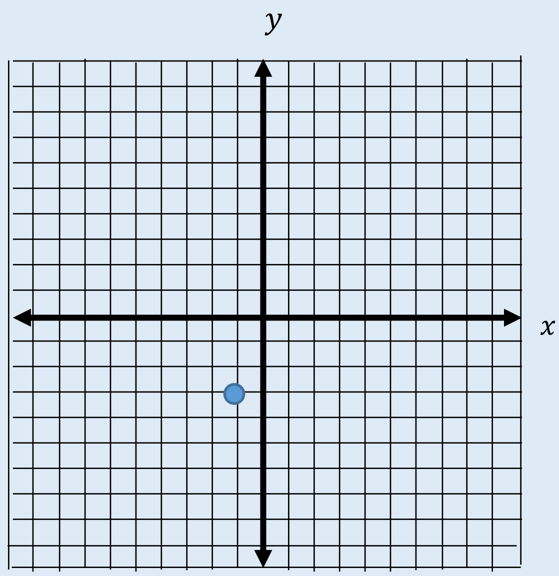
Be careful reading off the vertex because $h = -1$ and $k = -3$. Notice that we still take the opposite sign of what is inside the parentheses and the same sign for what is outside the parentheses.

vertex: $(-1, -3)$

axis of symmetry: $y = -3$

We also note that this one will open to the right since $a = 1$.

We can start graphing by plotting the vertex:

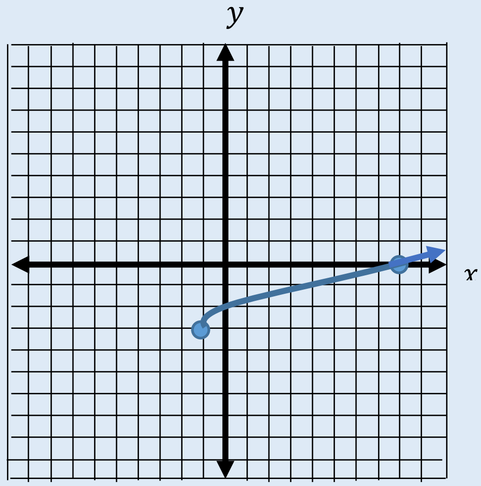


Now decide what you want to plug in next, but for y instead of x this time. We will choose to plug in 0 for y (since that will also give us the x-intercept at the same time).

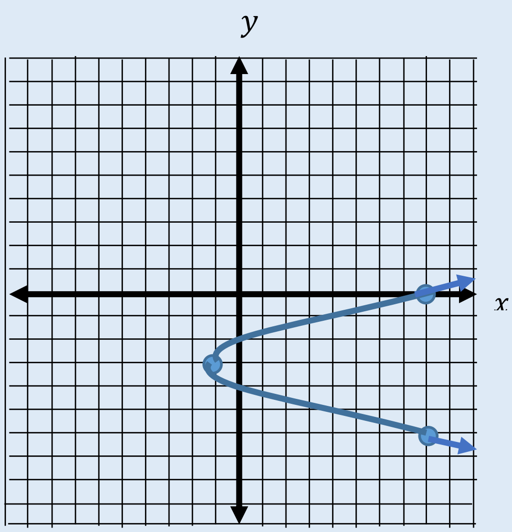
Plugging in 0 for y in the original equation gives us:

$$x = (0)^2 + 6 \cdot 0 + 8 = 8$$

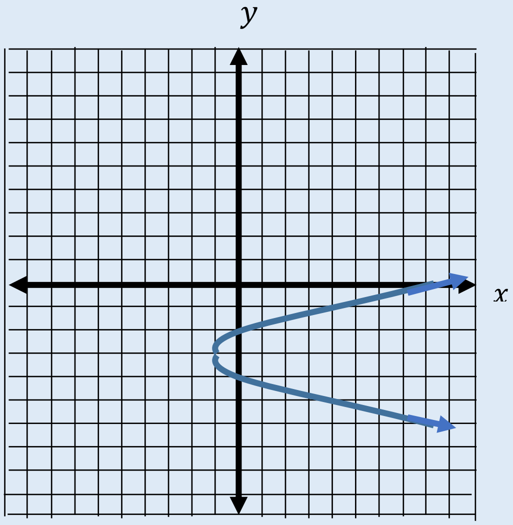
This means we have the point $(8,0)$ on the graph as well.



Now to get the other side of the parabola, use symmetry. Since you traveled a distance of 3 from the vertex to get to 0, you can go a distance of 3 in the other direction to land at -6 and you should get the same x-value. That means the point $(8, -6)$ will also be on the graph.



So our final graph (without the exaggerated points on it) looks like this:



We could also have used our value of $a = 1$ to move over to the right by 1 and then up and down 1 in to arrive at two points on the parabola.

The y-intercepts are fairly easy for this one if we notice the original equation factors when we set $x = 0$:

$$0 = y^2 + 6y + 8$$

$$\Rightarrow \quad 0 = (y + 4)(y + 2)$$

$$\Rightarrow \quad y = -4; \; y = -2$$

We might also be able to see them from the graph, but it is better to use algebra to check, since you might be off by a hair (e.g. .01) and you can't tell from just looking.

Note: You could also solve the completed square form of the equation and arrive at these answers rather quickly, too.

Now, let's write down the requested information:

vertex: $(-1, -3)$

axis of symmetry: $y = -3$

domain: $[-1, \infty)$

range: $(-\infty, \infty)$

x-intercept(s): $(8,0)$

y-intercept(s): $(0, -2), (0, -4)$

5. $x = -2y^2 + 4y + 1$

This example is not in graphing form, so we first need to complete the square to put it in graphing form:

$$\Rightarrow \quad x = -2\left(y^2 - 2y - \frac{1}{2}\right)$$

$$\Rightarrow \quad x = -2\left(y^2 - 2y + \mathbf{1} - \mathbf{1} - \frac{1}{2}\right)$$

$$\Rightarrow \quad x = -2\left[(y-1)^2 - \frac{3}{2}\right]$$

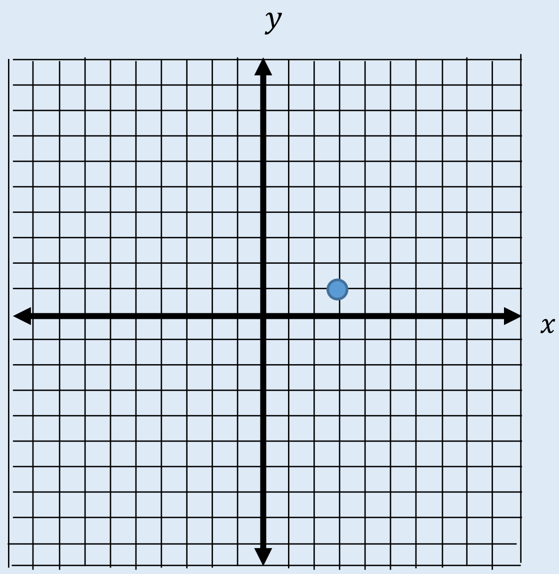
$$\Rightarrow \quad x = -2(y-1)^2 + 3$$

vertex: $(3,1)$

axis of symmetry: $y = 1$

We also note that this one will open to the left since $a = -2$.

We can start graphing by plotting the vertex:

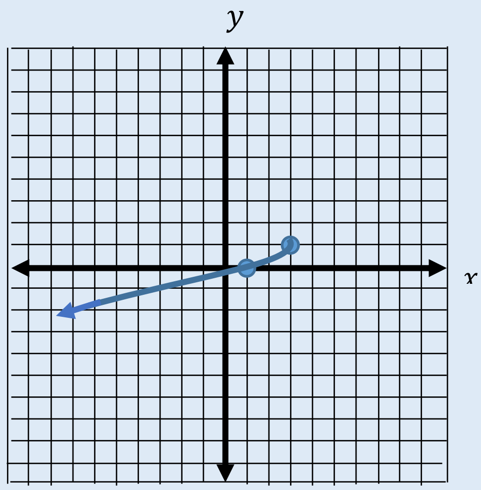


Now decide what you want to plug in next, but for y instead of x this time. We will choose to plug in 0 for y (since that will also give us the x-intercept at the same time).

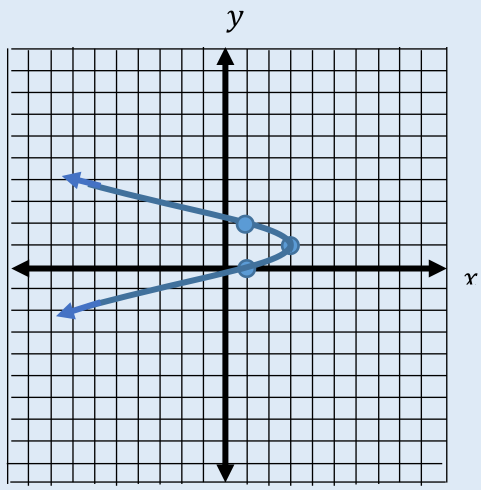
Plugging in 0 for y in the original equation gives us:

$$x = -2(0)^2 + 4 \cdot 0 + 1 = 1$$

This means we have the point $(1,0)$ on the graph as well.



Now to get the other side of the parabola, use symmetry. Since you traveled a distance of 1 from the vertex to get to 0, you can go a distance of 1 in the other direction to land at 2 and you should get the same x-value. That means the point $(1,2)$ will also be on the graph.



The y-intercepts are easier computed for this one by using the equation that has the square completed, although you could use the quadratic formula as well:

$$0 = -2(y-1)^2 + 3$$

⇨ $$-3 = -2(y-1)^2$$

⇨ $$\frac{3}{2} = (y-1)^2$$

⇨ $$\pm\sqrt{\frac{3}{2}} = y - 1$$

⇨ $$1 \pm \sqrt{\frac{3}{2}} = y$$

Rationalizing the denominators:

⇨ $$y = 1 \pm \frac{\sqrt{6}}{2}$$

or as decimal approximations $y = 2.225, y = -0.225$

Now, let's write down the requested information:

vertex: $(3,1)$

axis of symmetry: $y = 1$

domain: $(-\infty, 3]$

range: $(-\infty, \infty)$

x-intercept(s): $(1,0)$

y-intercept(s): $\left(0, 1 + \frac{\sqrt{6}}{2}\right), (0, 1 - \frac{\sqrt{6}}{2})$