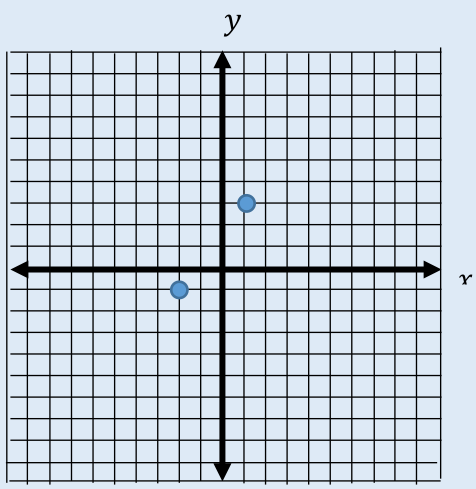
5.2 Graphing Lines; Slope of a Line

Any line can be determined by two points. If you know any two points on the line, you can get the entire line (due to the fact that lines are straight). This is only true for lines, not for other graphs.

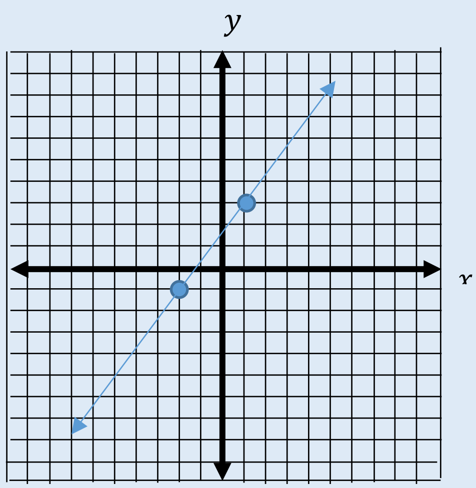
So, one way to graph a line is to use two points and connect them, provided that you have two points!

Example

Given the points $(1,3)$ and $(-2,-1)$, we can graph the line containing these points.



First, graph each of the points given.



Then connect the two points with a straight line. Now, you have all of the points on the line!

What if you are given an equation of the line rather than two points?

Example

Graph $y = x + 2$

We can make a table of values to determine points that we can use to graph the line. We can find two points by plugging values in for either x or y. Then, the values in the table give us ordered pairs (x, y) which are points that we can plot on the graph. Once we have two points plotted on the graph, we can "connect" the two points with a line which completes our graph. (Remember, this is only true for lines! For any other type of graph, two points would not be enough!)

To make a table of values, choose a number for either x or y. Then substitute the value into the given equation and solve. We can choose any value that we want for x since the domain of any (non-vertical) line is all real numbers.

Let's choose $x = 0$ since it is so easy to plug in.

$y = x + 2$	Substitute 0 for x, then solve for y.
$y = 0 + 2$	
$y = 2$	The corresponding ordered pair is $(0,2)$. Note: When you choose $x = 0$, the corresponding ordered pair happens to be the y-intercept (the point where the graph hits the y-axis).

Now, let's choose $y = 0$ since that is also easy to plug in.

$y = x + 2$	Substitute 0 for y, then solve for x.

$$\begin{array}{r} 0 = x + 2 \\ \underline{-2 \qquad -2} \end{array}$$

$-2 = x$ The corresponding ordered pair is $(-2,0)$. Note: When you choose $y = 0$, the corresponding ordered pair is the *x*-intercept (the point where the graph hits the *x*-axis).

.

Let's calculate a couple other points just for fun. If the given equation is in slope-intercept form (solved for y), it is generally easier to choose values for *x* instead of *y*.

Let's choose $x = 1$.

$y = x + 2$ Substitute 1 for *x*, then solve for *y*.

$y = 1 + 2$

$y = 3$ The corresponding ordered pair is $(1,3)$.

Let's choose $x = 3$.

$y = x + 2$ Substitute 3 for *x*, then solve for *y*.

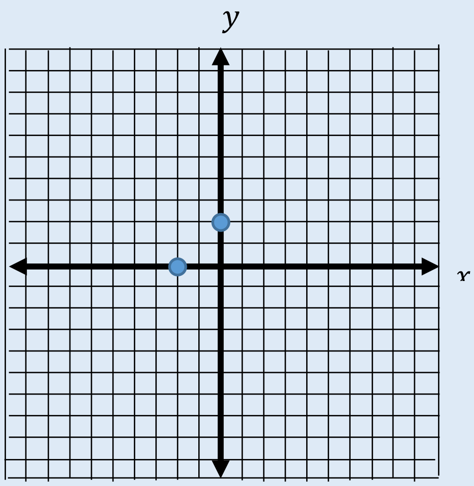
$y = 3 + 2$

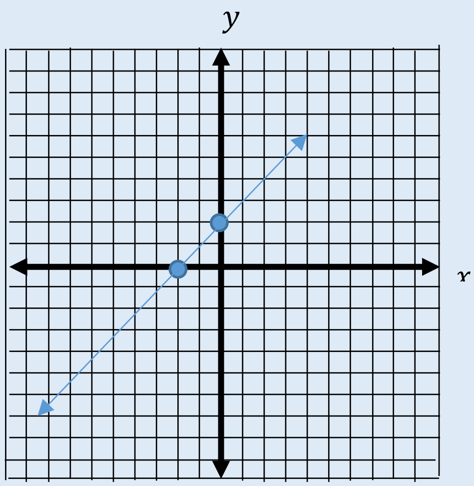
$y = 5$ The corresponding ordered pair is $(3,5)$.

Below is the table of values that corresponds to the example above.

x	***y***
0	2
-2	0
1	3
3	5

Now, we are ready to graph the line. We will use two of the points that we determined to graph the line. We can use the other two points from the table to check (these points should lie on the line that was graphed).





Let's choose the points $(0,2)$ and $(-2,0)$ and plot them, although we could use any two of the four points we found above.

Now, connect the two points and you will get the line. You will notice that the points $(1,3)$ and $(3,5)$ also lie on the line.

We can also state the domain and the range from our graph. The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$. This will be true of most lines, all but the horizontal and vertical lines that we will talk about in a little bit….

<u>Slope</u>: The slope of a line tells us how steep the line is. (It can also tell us about the rate of change of many real life phenomena and is hence the subject of the first semester of calculus - called the derivative or rate of change).

Conceptually:

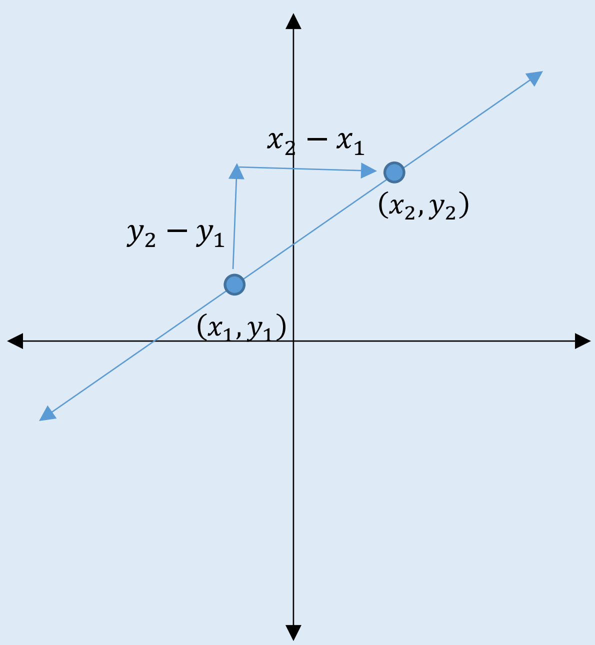
$$slope = \frac{rise}{run} = \frac{\uparrow}{\rightarrow} = \frac{\Delta y}{\Delta x} = \frac{change\ in\ y}{change\ in\ x}$$

Practically (used for computation):

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given two points, we can find the slope of the line using this formula.

Consider the graph below for a visual depiction of slope:



If you grab any two points on a line, call them (x_1, y_1) and (x_2, y_2), then the ratio of their differences, $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same!

Imagine this line getting flatter (horizontal). What would happen to the slope? What if the line became as steep as it could (vertical)? What could you say about the slope in that case?

We can also use the slope to graph a line as long as we have at least one point. If we start at the point and use the slope to get to another point, then we will have two points to connect! We will do this soon, but first we should get used to the formula, so let's calculate some slopes.

Examples

Find the slope of the line passing through the following points:

1. $(-1,3)$ and $(5,-7)$

Label the points. $(x_1, y_1) = (-1,3)$ and $(x_2, y_2) = (5,-7)$

Now, substitute the values into the slope equation and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-3}{5-(-1)} = \frac{-7-3}{5+1} = \frac{-10}{6} = \frac{-5}{3}$$

Note: We could have switched the “labels” on the points. We still get the same value for the slope.

$$(x_1, y_1) = (5,-7) \text{ and } (x_2, y_2) = (-1,3)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-7)}{-1-5} = \frac{3+7}{-1-5} = \frac{10}{-6} = \frac{-5}{3}$$

2. $(-3,4)$ and $(2,4)$

Label the points. $(x_1, y_1) = (-3,4)$ and $(x_2, y_2) = (2,4)$

Now, substitute the values into the slope equation and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-4}{2-(-3)} = \frac{0}{5} = 0$$

Note: When the slope is 0, that means the line is not steep at all. A line that is not steep at all is called a **horizontal** line. When the *y*-values are the same, this will always happen!

3. $(-3,4)$ and $(-3,7)$

Label the points. $(x_1, y_1) = (-3,4)$ and $(x_2, y_2) = (-3,7)$

Now, substitute the values into the slope equation and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{-3-(-3)} = \frac{7-4}{-3+3} = \frac{3}{0} \Longrightarrow \text{slope is undefined}$$

Note: A line that has an undefined slope is infinitely steep (or as steep as you can get), so it must be a **vertical** line. When the *x*-values are the same, this will inevitably happen!

4. $\left(\frac{-1}{2}, 4\right)$ and $\left(3, \frac{-5}{3}\right)$

Label the points. $(x_1, y_1) = \left(\frac{-1}{2}, 4\right)$ and $(x_2, y_2) = \left(3, \frac{-5}{3}\right)$

Now, substitute the values into the slope equation and simplify.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{-5}{3} - 4}{3 - \left(\frac{-1}{2}\right)} = \frac{\frac{-5}{3} - 4}{3 + \frac{1}{2}} = \frac{\frac{-5}{3} - \frac{4}{1} \cdot \frac{3}{3}}{\frac{3}{1} \cdot \frac{2}{2} + \frac{1}{2}} = \frac{\frac{-5}{3} - \frac{12}{3}}{\frac{6}{2} + \frac{1}{2}} = \frac{\frac{-17}{3}}{\frac{7}{2}} = \frac{-17}{3} \cdot \frac{2}{7} = \frac{-34}{21}$$

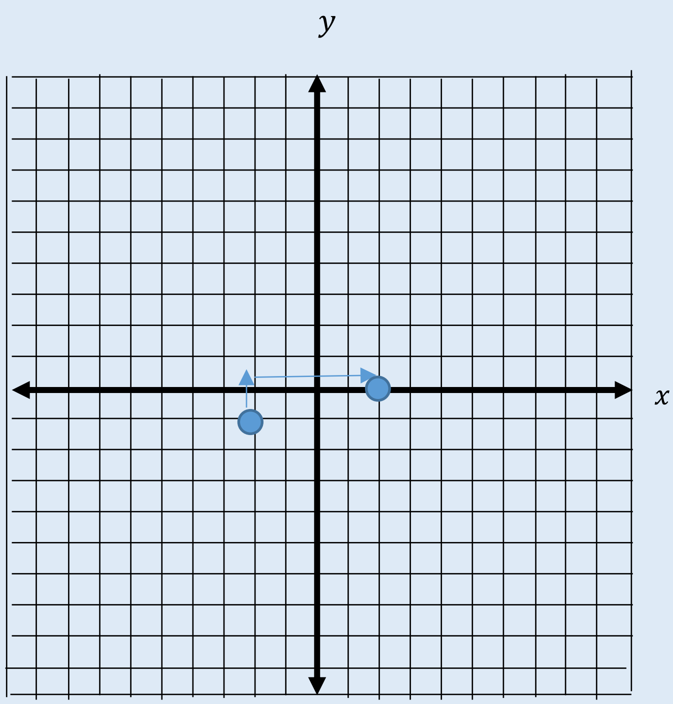
Now, we can graph a line if we are given a point and a slope.

We will begin by plotting the given point. From the point that is plotted on the graph, we will move based on the slope. The numerator of the slope is the “rise” or the up/down movement (a positive value will move up or a negative value will move down) and the denominator of the slope is the “run” or the left/right movement (a positive value will move right or a negative value will move left). If your slope is negative, you can choose to give the negative sign to either the numerator or the denominator, but I think it makes more sense to always give it to the numerator because then you only have to worry about going up (positive) or down (negative), but you will ALWAYS go to the right (since the denominator will always be positive). After moving based on the slope, we will plot a point at the place we end up. Now, connect the two plotted points to get the graph of the line.

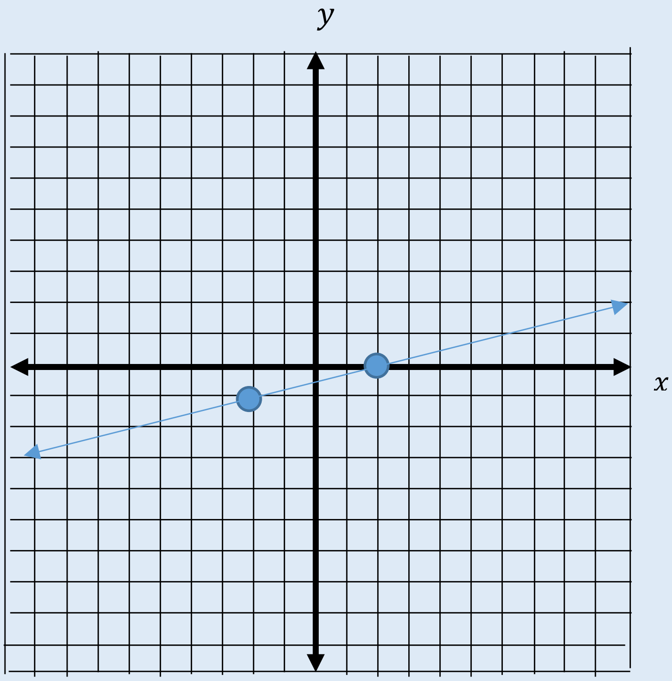
Examples

1. Graph the line given the point $(-2,-1)$ with slope $m = \frac{1}{4}$.

   For $m = \frac{1}{4} = \frac{\uparrow 1}{\rightarrow 4}$, we will move up 1 and right 4.



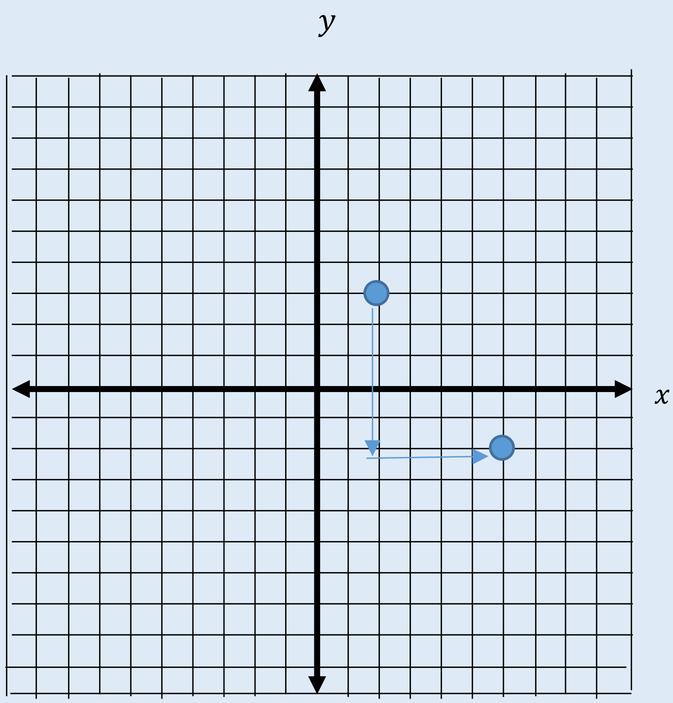
From the given point $(-2,-1)$, move up 1 and to the right 4 and you will land at the point $(2,0)$.



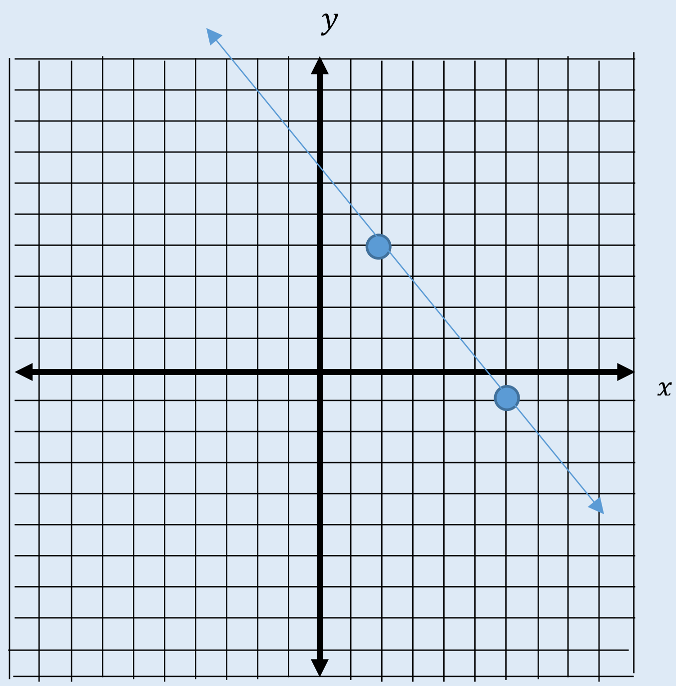
Now connect your points and you have your line.

2. Graph the line that passes through the point $(2,3)$ with slope $m = \frac{-5}{4}$.

For $m = \frac{-5}{4} = \frac{\downarrow 5}{\rightarrow 4}$, we will move down 5 and right 4.



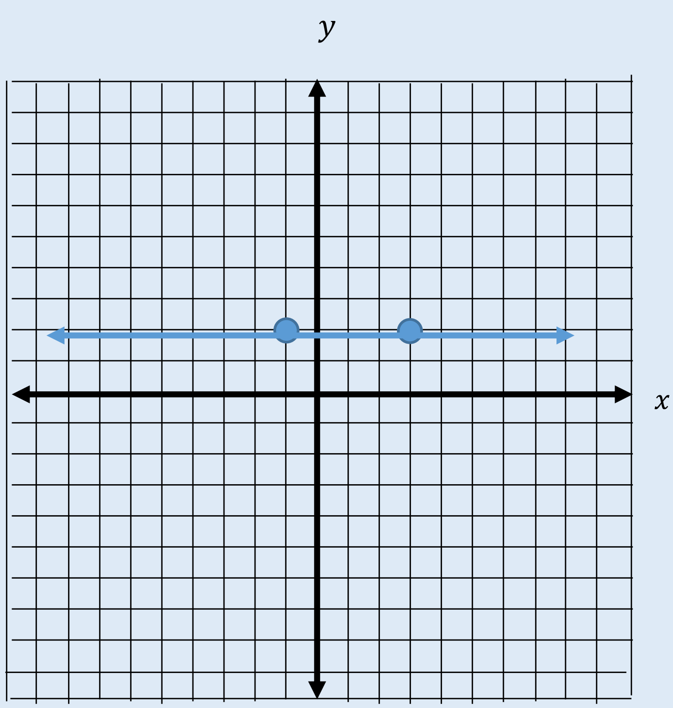
From the given point $(2,3)$, move down 5 and to the right 4 and you will land at the point $(-2,6)$.



Now connect your points and you have your line.

Horizontal and Vertical Lines

Horizonal Lines

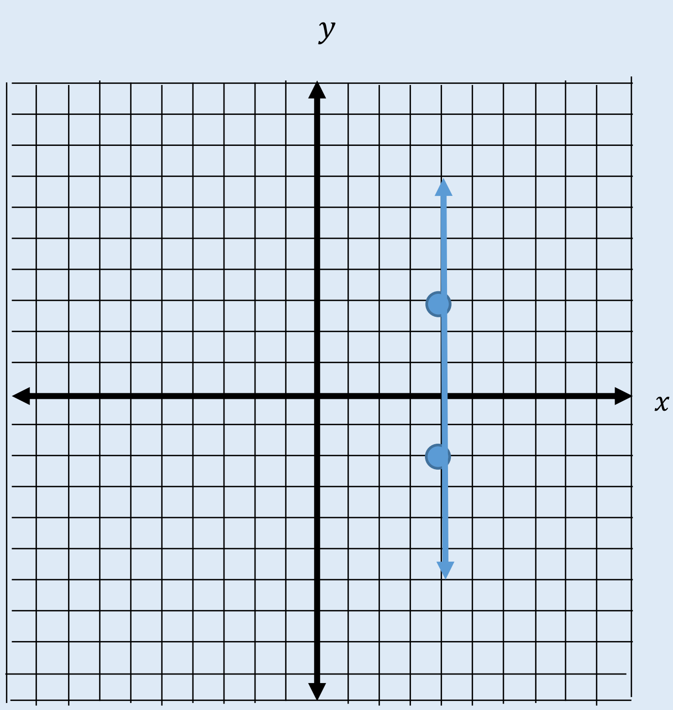


If we consider the horizontal line shown in the graph above and we grab any two points off of the line to calculate the slope, for example $(-1,2)$ and $(3,2)$, then we will always get 0 in the numerator since the *y*-values are the same. This is why every horizontal line has a slope of 0.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{3-(-1)} = \frac{0}{4} = 0$$

Notice that horizontal lines always have the same *y* value at every point and that *y* does not depend on *x*. For the example above, it doesn't matter what *x* is, *y* is always 2, so the equation of that horizontal line is $y = 2$. In fact, the equation of any horizontal line looks like this: $y = number$ (there is no *x* involved) and that is how you can identify a horizontal line from its equation!

Vertical Lines



If we consider the vertical line shown in the graph above and we grab any two points off of the line to calculate the slope, for example $(4,-2)$ and $(4,3)$, then we will always get 0 in the denominator since the *x*-

values are the same. This is why every vertical line has an undefined slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-2)}{4-4} = \frac{3+2}{4-4} = \frac{5}{0} \Longrightarrow \text{slope is undefined}$$

Notice that vertical lines always have the same *x* value at every point and that *x* does not depend on y. For the example above, it doesn't matter what y is, *x* is always 4, so the equation of that vertical line is $x = 4$. In fact, the equation of any vertical line looks like this: $x = number$ (there is no *y* involved) and that is how you can identify a vertical line from its equation!

Slope-Intercept Form of a Line

If the linear equation is solved for *y*, then it is in **slope-intercept form**, which means we can read the slope and the y-intercept right off of the equation:

$$y = mx + b$$

slope   *y*-intercept $(0, b)$

Examples

Find the slope and *y*-intercept given the equation.

1. $y = \frac{2}{3}x + 5$

$m = \frac{2}{3}$ *y*-intercept: $(0,5)$

2. $3x - 2y = 4$

Solve for y first.

$$\begin{array}{rl} 3x - 2y &= 4 \\ \underline{-3x \qquad\quad} & \underline{\; - 3x \;} \\ -2y &= -3x + 4 \end{array}$$

$$\frac{-2y}{-2} = \frac{-3}{-2}x + \frac{4}{-2}$$

$$y = \frac{3}{2}x - 2$$

$m = \frac{2}{3}$ $\qquad$ y-intercept: $(0, -2)$

3. $y - 7 = 0$

$y = 7 \Longrightarrow$ horizontal line (since $y = 0x + 7$)

$m = 0$ $\qquad$ y-intercept: $(0,7)$

4. $x - 7 = 0$

$x = 7 \Longrightarrow$ vertical line

m is undefined and no y-intercept

We can graph very easily from an equation that is in slope-intercept form because we have a point to begin with (the y-intercept) and a slope to get us to another point.

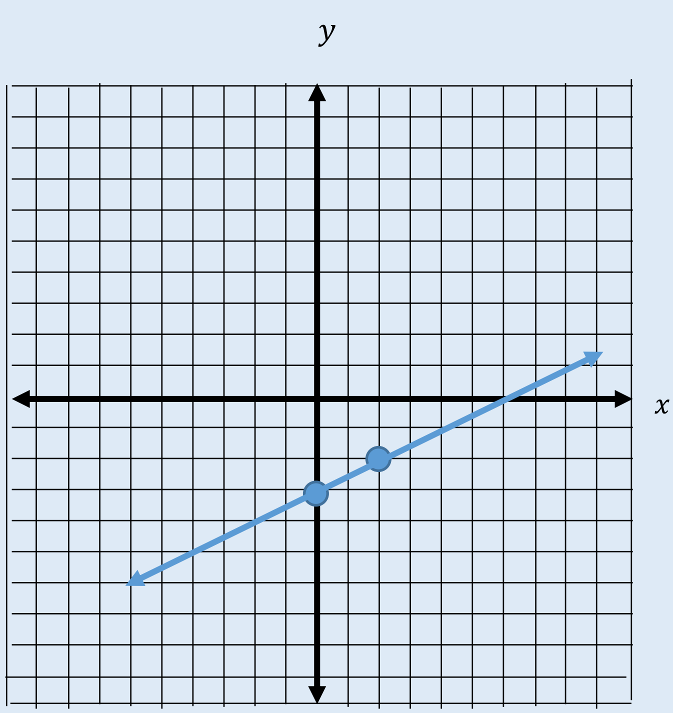
Examples

1. Graph the line $y = \frac{1}{2}x - 3$

We can graph this the same way we graphed a line given the slope and a point. Since the equation is written in slope-intercept form, we can identify the slope and the *y*-intercept (which gives us a point and the slope).

$m = \frac{1}{2}$ *y*-intercept: $(0, -3)$

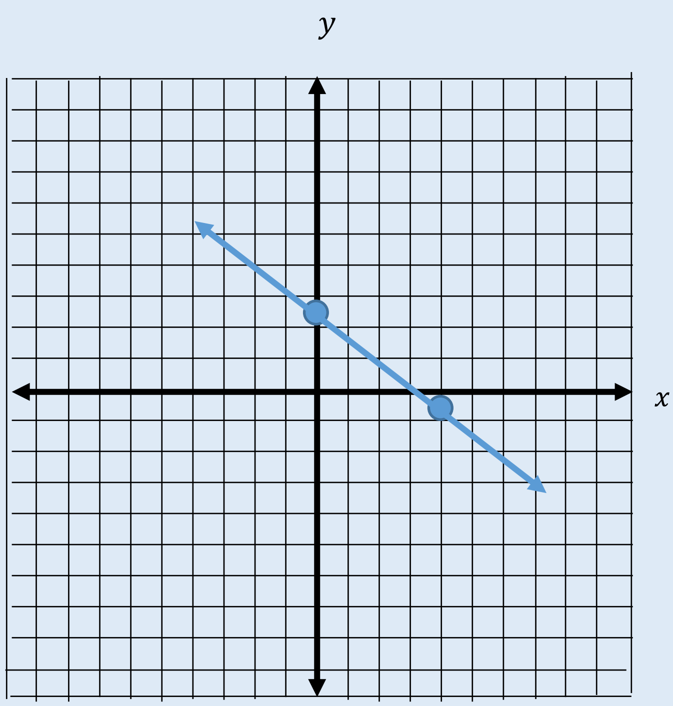
Begin by plotting the *y*-intercept on the graph. From that point on the graph, "move" according to the slope (up 1, to the right 2) to get a second point on the graph. Then connect the two points with a line to complete the graph.



2. Graph the line $f(x) = \frac{-3}{4}x + \frac{5}{2}$

We can rewrite the equation of the line as $y = \frac{-3}{4}x + \frac{5}{2}$.

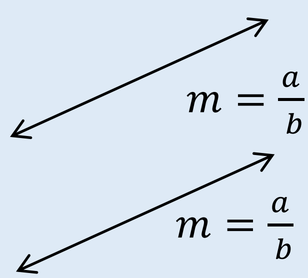
$m = \frac{-3}{4}$ y-intercept: $\left(0, \frac{5}{2}\right)$



Parallel and Perpendicular Lines

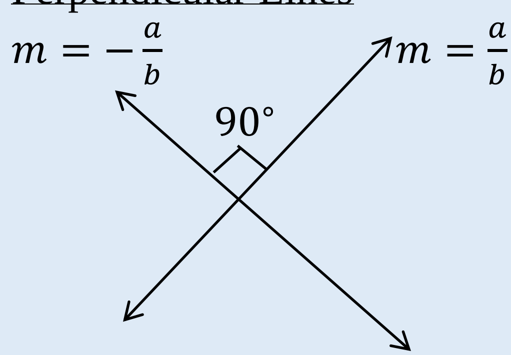
Parallel lines have the same slope and perpendicular lines have negative reciprocal slopes.

Parallel Lines



same slope

Perpendicular Lines



Examples

Determine whether the following lines are parallel, perpendicular, or neither.

1. $y = \frac{1}{2}x - 3$ and $2y - x = 1$

$m = \frac{1}{2}$

$$\begin{array}{rcl} 2y - x &=& 1 \\ \underline{+x} && \underline{+x} \\ 2y &=& x + 1 \end{array}$$

$$\frac{2y}{2} = \frac{1}{2}x + \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

$$m = \frac{1}{2}$$

Since both lines have the same slope and different *y*-intercepts, the lines are parallel.

2. $-2x + 10y = 7$ and $5x + y = 3$

$$\begin{array}{rcl} -2x + 10y &=& 7 \\ \underline{+2x} && \underline{+2x} \\ 10y &=& 2x + 7 \end{array}$$

$$\frac{10}{10}y = \frac{2}{10}x + \frac{7}{10}$$

$$y = \frac{1}{5}x + \frac{7}{10}$$

$$m = \frac{1}{5}$$

$$\begin{array}{rcl} 5x + y &=& 3 \\ \underline{-5x} && \underline{-5x} \\ y &=& -5x + 3 \end{array}$$

$$m = -5$$

Since the slopes of the two lines are negative reciprocals (different signs and "flipped"), the lines are perpendicular.

Note: We can also determine that two lines are perpendicular if the product of their slopes is -1.

$$\frac{1}{5} \cdot \frac{-5}{1} = -1$$

3. $y = \frac{2}{3}x - 5$ and $3x - 2y = 7$

$$\begin{aligned} 3x - 2y &= 7 \\ \underline{-3x \qquad} &\underline{\;\; - 3x} \\ -2y &= -3x - 7 \end{aligned}$$

$m = \frac{2}{3}$

$$\frac{-2}{-2}y = \frac{-3}{-2}x - \frac{7}{-2}$$

$$y = \frac{3}{2}x + \frac{7}{2}$$

$m = \frac{3}{2}$

Since the slopes are not the same, the lines are not parallel and since the slopes are not negative reciprocals, the lines are not perpendicular. Therefore, the lines are neither parallel or perpendicular.