4.5 Polynomial and Rational Inequalities

We know how to solve linear inequalities, like $3x - 4 > 0$, but what about non-linear polynomial inequalities such as $x^2 - 3x - 4 > 0$?

When solving a non-linear polynomial equation, we used the zero product property: $a \cdot b = 0\ if\ and\ only\ if\ either\ a = 0\ or\ b = 0$. So we were able to solve $x^2 - 3x - 4 = 0$ by simply factoring it and setting each factor equal to 0. But for inequalities, we have no such property. If we have $a \cdot b > 0$, that doesn't mean that either $a > 0\ or\ b > 0$. Consider, for example, the product $(-3)(-5)$. Clearly this product is positive (greater than 0), but neither of the factors are greater than 0.

One way we could approach this problem is to break it into factors and use compound inequalities, but that process can get quite tedious if you have more than two factors. For example, if we have three factors $a \cdot b \cdot c > 0$, think about how many possibilities could make this true. If any two of the three are negative and the other one is positive, the result will be positive. We would have to consider each of the following compound inequalities and then combine them with or's between them:

$a > 0\ and\ b > 0\ and\ c > 0$
Or
$a < 0\ and\ b < 0\ and\ c > 0$
Or
$a > 0\ and\ b < 0\ and\ c < 0$
Or
$a < 0\ and\ b > 0\ and\ c < 0$

Just imagine that $a, b, and\ c$ represent linear factors like $3x + 5$. This could get very tedious, but there is a better way to organize all of this information in a chart. We will introduce this method in the following examples.

<u>Examples</u>

Solve each of the following inequalities and write the solution in interval notation.

1. $x^2 - 3x - 4 > 0$

$(x - 4)(x + 1) > 0$

First, let's approach this problem using compound linear inequalities to get a better understanding:

The product of two numbers is positive (> 0) if both numbers are positive or if both numbers are negative, so treating our factors above as numbers, we have two possibilities:

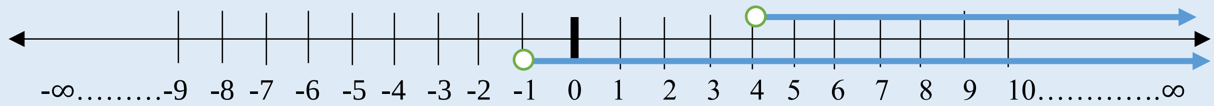
$(x - 4 > 0\ and\ x + 1 > 0)$ or $(x - 4 < 0\ and\ x + 1 < 0)$

Solve each compound inequality separately and then combine them with the "or" as follows:

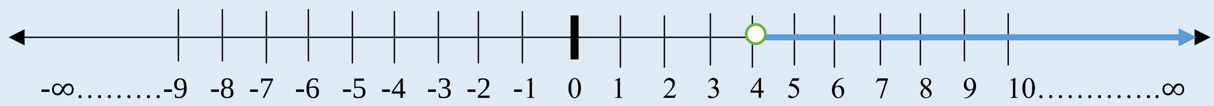
Solving the first compound inequality:

$$x - 4 > 0\ and\ x + 1 > 0$$

⇨ $x > 4\ and\ x > -1$



Taking the intersection of the two, we get:

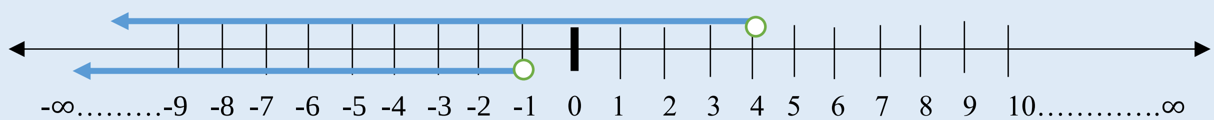


⇨ $(4, \infty)$

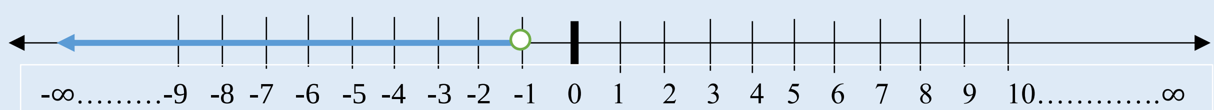
Now, solving the other compound inequality:

$$x - 4 < 0 \; and \; x + 1 < 0$$

⇨ $x < 4 \; and \; x < -1$



Taking the intersection of the two, we get:



⇨ $(-\infty, -1)$

Now, combining the two results with an "or", we get:

$$(-\infty, -1) \cup (4, \infty)$$

If we had more than two compound inequalities, this process could get very long. There is another way that will summarize these results within a single chart. This chart method is also used in calculus to analyze “functions” and their “derivatives” and is often also used for graphing rational functions. It is a method that is worth learning because you will use it on more than one occasion.

Let’s try the same problem using a sign chart:

$x^2 - 3x - 4 > 0$

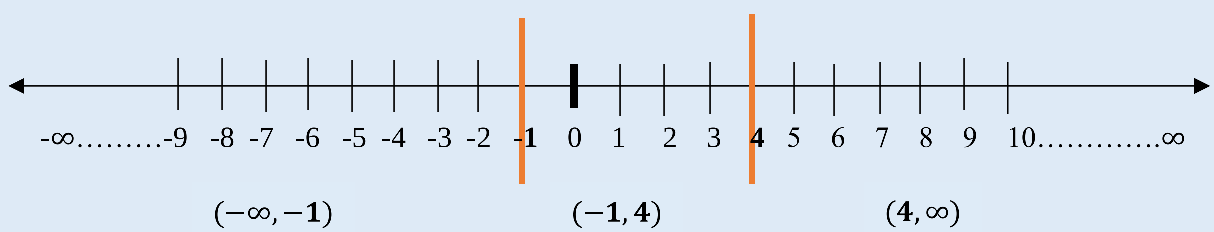
First factor the polynomial as usual:

$(x - 4)(x + 1) > 0$

Set each factor equal to 0 to get the *critical values* (Critical values are the values at which the sign of the factor will change from positive to negative or vice versa). For example, at 4, the factor $x - 4$ changes sign since on one side of 4, we get negatives when we plug in values (such as 3.9 *or* 3) and on the other side of 4, we get positives when we plug in values (like 4.1 *or* 5 for example).

Critical values: -1 *and* 4

We will use the critical values to break up the number line into test intervals. (For this reason, we should put them in order from smallest to largest.)



The test intervals are $(-\infty, -1), (-1, 4), (4, \infty)$

To set up your chart, make a table with the factors down the first column and the test intervals in the first row as follows:

	$(-\infty,-1)$	$(-1,4)$	$(4,\infty)$
$x-4$			
$x+1$			
Result:			

Now, for each test interval, we will choose *any* value inside of the test interval (it has to be inside – NOT an endpoint since those give you 0) and plug it into each factor to see if we get a positive or a negative number. We only care about the sign, since we multiply these signs together to get our result on each interval. (Any value that you choose in that interval will give you the same result because there is nowhere inside of the interval where the sign changes or else we would have another critical value inside of that interval!)

Test value:	-2	0	5
	$(-\infty,-1)$	$(-1,4)$	$(4,\infty)$
$x-4$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
Result:	$+$	$-$	$+$

Since our original inequality was >0, we want the intervals that have a positive result. So the solution is: $(-\infty,-1)\cup(4,\infty)$.

If our original inequality had been <0, then we would have taken the interval that gave us the negative result to be our solution.

2. $x^2 - 8x < -15$

First bring everything to one side to get 0 on the other side:

$x^2 - 8x + 15 < 0$

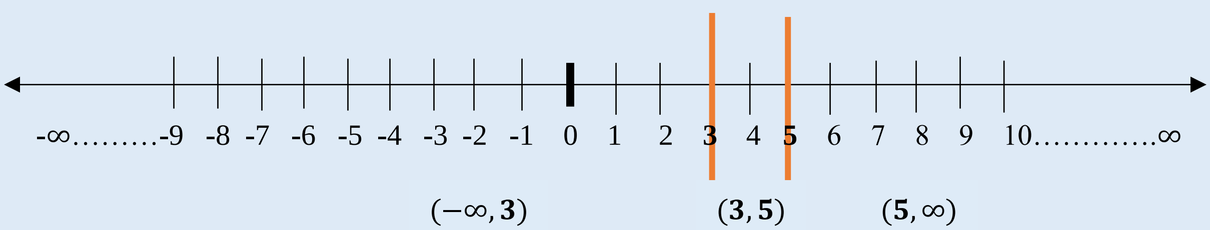
Note that < 0 means negative, so we will be looking for negative results in our sign chart.

Now, factor and get your critical values:

$(x - 3)(x - 5) < 0$

Critical values: $3 \textit{ and } 5$

Use the critical values to break up the number line into test intervals:



The test intervals are $(-\infty, 3), (3,5), (5, \infty)$

Now set up your chart:

	$(-\infty, 3)$	$(3,5)$	$(5, \infty)$
$x - 3$			
$x - 5$			
Result:			

Let’s fill in the chart now by plugging in test values from inside of each interval (you can choose a different test value than we did to see that you will get the same result):

Test value:	0	4	6
	$(-\infty, 3)$	$(3,5)$	$(5, \infty)$
$x-3$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
Result:	$+$	$-$	$+$

A note about endpoints:

If our original inequality had been ≤ 0 instead of just < 0, then we would have included our endpoints in the answer because the endpoints actually make the polynomial equal to 0. So if our problem had been $x^2 - 8x + 15 \leq 0$, then our answer would have been $[3,5]$ instead of $(3,5)$.

3. $x^2 + 6x \geq -9$

First bring everything to one side to get 0 on the other side:

$$x^2 + 6x + 9 \geq 0$$

Note that ≥ 0 means positive, so we will be looking for positive results in our sign chart.

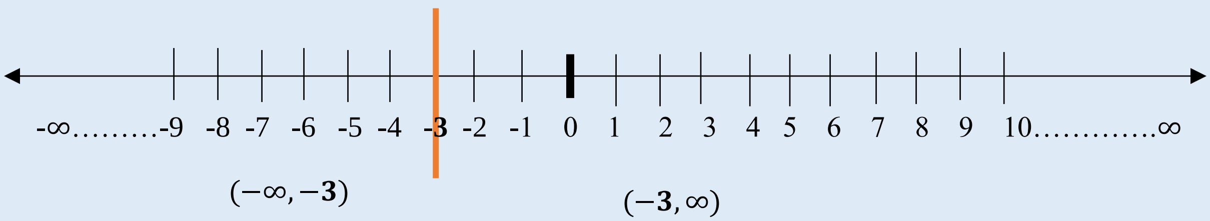
Now, factor and get your critical values:

$(x+3)(x+3) \geq 0$

Critical values: -3

Note that you could use logic to find the solution rather quickly if you recognize that you have a perfect square: $(x+3)^2 \geq 0$ and this will always be true for any real number x. Therefore the solution is "All real numbers" or $(-\infty, \infty)$.

Use the critical values to break up the number line into test intervals:



The test intervals are $(-\infty, -3), (-3, \infty)$

Make sure to include both factors in your chart because their signs need to be multiplied together to get the sign of their product:

	$(-\infty, -3)$	$(-3, \infty)$
$x+3$		
$x+3$		
Result:		

Let's fill in the chart now by plugging in test values from inside of each interval (you can choose a different test value than we did to see that you will get the same result):

Test value: -4 0

	$(-\infty,-3)$	$(-3,\infty)$
$x+3$	$-$	$+$
$x+3$	$-$	$+$
Result:	$+$	$+$

So if this problem were >0, then our solution would be $(-\infty,-3)\cup(-3,\infty)$, which means "everything *except* -3". But the fact that we have the "equal to symbol" included in our symbol ≥ 0 means that we will include the endpoints as well, so our solution becomes:

$$(-\infty,-3]\cup[-3,\infty)$$

But this can be simplified to be $(-\infty,\infty)$.

We now know how to solve polynomial inequalities (as long as we can factor the polynomial), but what about rational inequalities? The process for rational inequalities is almost identical because the same rules apply when multiply and dividing signs. For example, multiplying two negatives gives you a positive, but so does dividing two negatives. The sign chart will give us the same results if we just continue to multiply the signs as we have been doing and include the factors and critical values from the denominator in our chart. The only difference is that when you have the "equal to" symbol involved and you have to check the endpoints, you cannot include the endpoints that make the denominator equal 0 since those values make the rational expression undefined. Therefore, the only critical values that can be bracketed are the ones that come from the numerator.

4. $\frac{x^2-x-12}{x-1} < 0$

Note that < 0 means negative, so we will be looking for negative results in our sign chart.

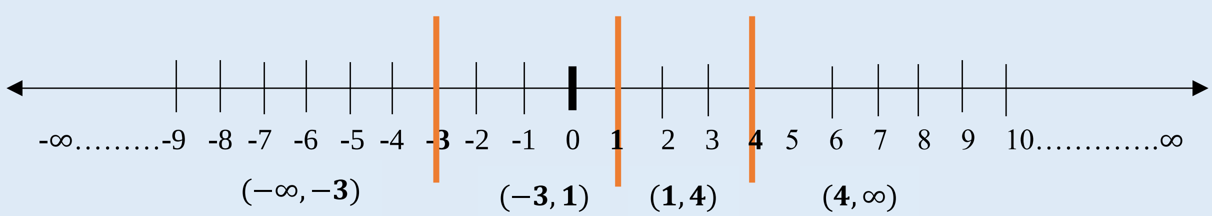
Now, factor and get your critical values:

$$\frac{(x-4)(x+3)}{x-1} < 0$$

Critical values: $4, -3 \; and \; 1$

Put these in order: $-3, 1, 4$

Use the critical values to break up the number line into test intervals:



The test intervals are $(-\infty, -3), (-3,1), (1,4), (4, \infty)$

Now set up your chart:

	$(-\infty,-3)$	$(-3,1)$	$(1,4)$	$(4,\infty)$
$x-4$				
$x+3$				
$x-1$				
Result:				

Now fill in the chart:

Test value: -4 0 2 5

	$(-\infty,-3)$	$(-3,1)$	$(1,4)$	$(4,\infty)$
$x-4$	$-$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
Result:	$-$	$+$	$-$	$+$

One thing to keep in mind as you are multiplying your signs together is that the positives don't really contribute anything (since multiplying by a positive never changes a sign) so you could just count the negatives to get the result. We know an even number of negatives multiplied together will always produce a positive, while an odd number of negatives will produce a negative.

Our solution to this inequality is: $(-\infty,-3)\cup(1,4)$

How would the solution be different if instead of < 0, this inequality had ≤ 0?

The only difference is that we would include the endpoints that make the numerator 0 (since when the numerator is 0, the entire fraction will be 0 because $\frac{0}{anything\ but\ 0} = 0$).

So our solution to $\frac{x^2-x-12}{x-1} \leq 0$ would be $(-\infty,-3] \cup (1,4]$.

Notice that the 1 does not get a bracket since it doesn't make this expression equal 0, but rather it makes the expression undefined.

Sometimes, you may be tempted to cross multiply when you are working with inequalities, but you cannot do that if there are any variables in the denominator. This is because you do not know what that variable represents and hence you do not know whether or not you are multiplying by a negative which would reverse the inequality. So instead you will need to subtract in order to get 0 on one side and then proceed to combine your fractions into a single rational expression. The next two examples illustrate this process.

5. $\frac{x}{x+9} \geq \frac{1}{x+1}$

First, we need to subtract $\frac{1}{x+1}$ from both sides to obtain

$$\frac{x}{x+9} - \frac{1}{x+1} \geq 0$$

Now combine them into a single rational expression by getting a common denominator:

⇨ $$\frac{x(x+1)}{(x+9)(x+1)} - \frac{1(x+9)}{(x+1)(x+9)} \geq 0$$

⇨ $$\frac{x(x+1)}{(x+9)(x+1)} - \frac{1(x+9)}{(x+1)(x+9)} \geq 0$$

⇨ $$\frac{x^2-9}{(x+9)(x+1)} \geq 0$$

Now factor the numerator:

⇨ $$\frac{(x+3)(x-3)}{(x+9)(x+1)} \geq 0$$

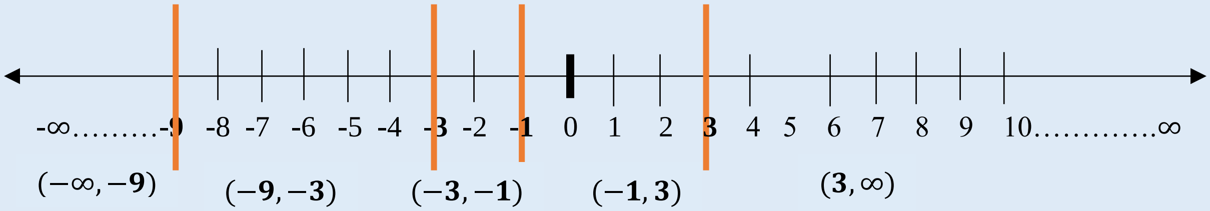
We are ready to proceed now.

Note that ≥ 0 means positive, so we will be looking for positive results in our sign chart.

You can get your critical values by setting each factor equal to 0, as usual: $-3, 3, -9 \; and - 1$

Put these in order: $-9, -3, -1, 3$

Use the critical values to break up the number line into test intervals:



The test intervals are
$(-\infty, -9), (-9, -3), (-3, -1), (-1, 3), (3, \infty)$

Test value: -10 -4 -2 0 4

	$(-\infty, -9)$	$(-9, -3)$	$(-3, -1)$	$(-1, 3)$	$(3, \infty)$
$x + 3$	$-$	$-$	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$-$	$-$	$+$	$+$
$x + 9$	$-$	$+$	$+$	$+$	$+$
Result:	$+$	$-$	$+$	$-$	$+$

The intervals we want to keep are: $(-\infty, -9) \cup (-3, -1) \cup (3, \infty)$

We have to include the endpoints that make the numerator equal to 0, so the solution to this inequality is:

$$(-\infty, -9) \cup [-3, -1) \cup [3, \infty)$$

6. $\frac{3}{x-2} \leq \frac{4}{x}$

First, we need to subtract $\frac{4}{x}$ from both sides to obtain

$$\frac{3}{x-2} - \frac{4}{x} \leq 0$$

Now combine them into a single rational expression by getting a common denominator:

⇨ $$\frac{3x}{x(x-2)} - \frac{4(x-2)}{x(x-2)} \leq 0$$

⇨ $$\frac{8-x}{x(x-2)} \leq 0$$

When you combine your numerators together to get $3x - 4(x-2) = 3x - 4x + 8 = -x + 8$, you might choose to factor out -1 and write it as $-(x-8)$. If you do this, make sure to include the factor of -1 in your sign chart because any negative numbers will affect the sign of the result. It is probably easier to leave it as $-x + 8 \; or \; 8 - x$ (since that is already linear so there is no need to factor)

We are ready to proceed now.

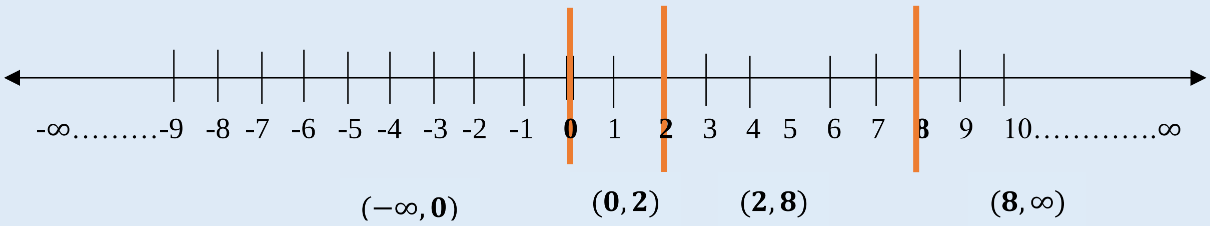
Note that ≤ 0 means negative, so we will be looking for negative results in our sign chart.

You can get your critical values by setting each factor equal to 0, as usual:

Critical values: $8, 0, and\ 2$

Put these in order: $0, 2, 8$

Use the critical values to break up the number line into test intervals:



The test intervals are $(-\infty, 0), (0,2), (2,8), (8, \infty)$

Test value: -1 1 3 9

	$(-\infty, 0)$	$(0,2)$	$(2,8)$	$(8, \infty)$
$8 - x$	$+$	$+$	$+$	$-$
x	$-$	$+$	$+$	$+$
$x - 2$	$-$	$-$	$+$	$+$
Result:	$+$	$-$	$+$	$-$

The intervals we want to keep are: $(0,2) \cup (8, \infty)$

We have to include the endpoints that make the numerator equal to 0, so the solution to this inequality is:

$$(0,2) \cup [8, \infty)$$