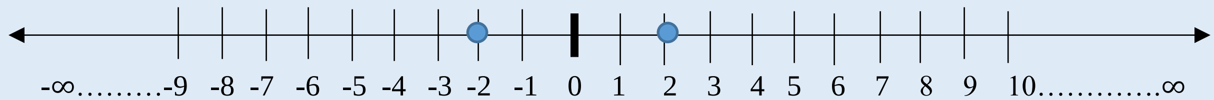
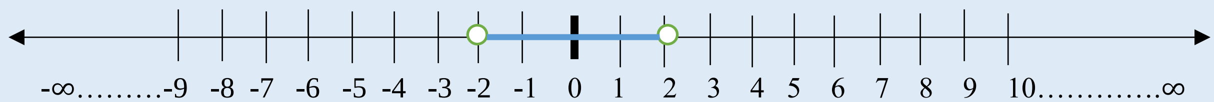
4.4 Solving Absolute Value Linear Inequalities

We understand absolute value equations pretty well. As a simple example, consider the equation $|x| = 2$. We know that there are two numbers that will make this true: If $x = 2$, then it is obviously true, but also if $x = -2$, then we can see it is also true. Graphing our solutions is not that interesting, as it just gives us two points on the number line, one at 2 and the other at -2:

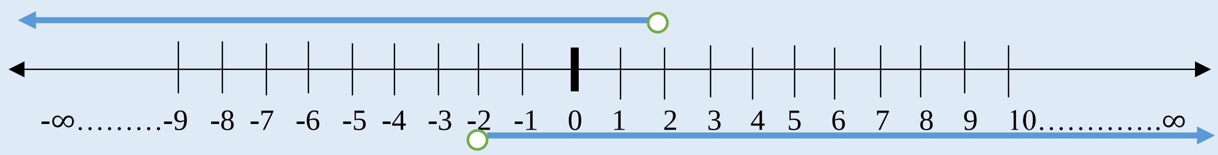


Let's now consider the inequality: $|x| < 2$.

Think about the numbers that will make this true…Is 3 a solution? Is 1.5 a solution? Is 0 a solution? Is -1.5 a solution? What about -3? If you carefully consider the numbers that will make this inequality true, you should arrive at all of the numbers between -2 and 2:



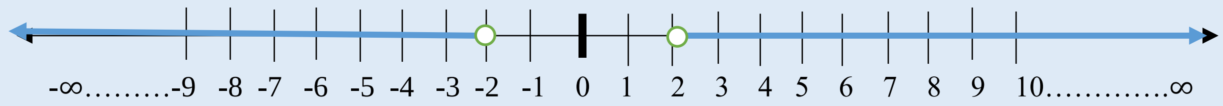
This should remind you of the "and" problems we just did in the last section since often the graphs are segments like this. This segment could indeed be thought of as the intersection of two pieces on the graph:



So the inequality $|x| < 2$ could also be written as the compound inequality $x < 2$ *and* $x > -2$. These are equivalent ways of expressing the same thing. We can split any absolute value inequality with a $<$ $or\ \leq$ symbol into two inequalities with an "and" between them in this way.

Similarly, consider the inequality $|x| > 2$.

Think about the numbers that will make this true…Is 3 a solution? Is 1.5 a solution? Is 0 a solution? Is -1.5 a solution? What about -3? If you carefully consider the numbers that will make this inequality true, you should arrive at all of the numbers smaller than -2 and greater than 2:



So the inequality $|x| > 2$ could be thought of as the compound inequality $x > 2\ or\ x < -2.$ We can split any absolute value inequality with a $>$ $or\ \geq$ symbol into two inequalities with an "or" between them in this way.

To summarize, when you are dealing with an absolute value inequality, first decide if it is an "and" or an "or" compound inequality by looking at the comparison symbol:

$\|\ \ \| <, \leq$	⇨	and
$\|\ \ \| >, \geq$	⇨	or

Then split into two inequalities with the appropriate logical connector "and" or "or" by simply removing the absolute value symbol for the first inequality and in addition reversing the comparison symbol and changing the sign of the number for the second inequality. We will do some examples that will clarify the process.

Examples

Solve each inequality for the indicated variable, graph the solution on a number line and write the answer in interval notation.

1. $|x + 2| \geq 5$

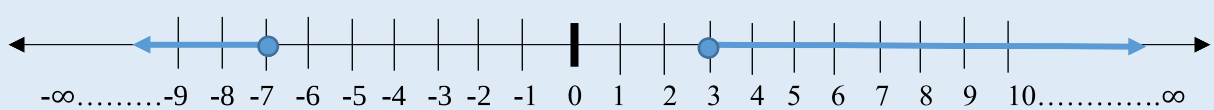
First, we must determine whether this is an "and" or an "or" situation and we can do that by looking at the comparison symbol. This is an "or" situation because it is "absolute value greater than (or equal to)":

$$\begin{array}{r} x + 2 \geq 5 \\ -2 \quad -2 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} x + 2 \leq -5 \\ -2 \quad -2 \\ \hline \end{array}$$

⇨ $x \geq 3$ ⇨ $x \leq -7$

So we have the solution " $x \geq 3$ or $x \leq -7$ ".

The graph is as follows:



Interval notation: $(-\infty, -7] \cup [3, \infty)$

2. $|x + 2| \leq 5$

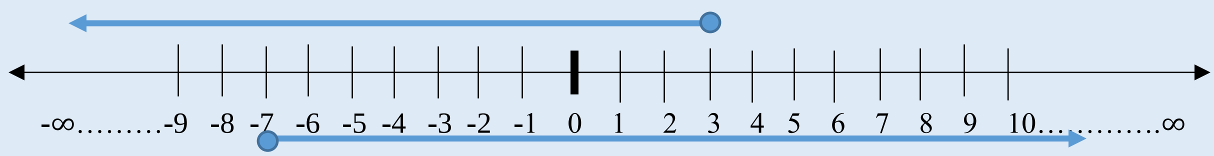
Again, we must determine whether this is an "and" or an "or" situation and we can do that by looking at the comparison symbol. This is an "and" situation because it is "absolute value less than (or equal to)":

$$x + 2 \leq 5 \qquad \text{and} \qquad x + 2 \geq -5$$
$$\underline{\quad -2 \quad -2} \qquad\qquad \underline{\quad -2 \quad -2}$$

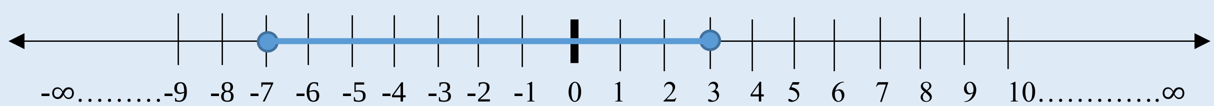
⇨ $x \leq 3$ ⇨ $x \geq -7$

So we have the solution " $x \leq 3$ and $x \geq -7$ ".

Graphing them both on the same number line, we get:



And taking the intersection:



Interval notation: $[-7,3]$

There is a quicker way to do the "and" problems, however. We could have written the problem in the beginning as a double inequality:

$$|x + 2| \leq 5$$

⇨ $-5 \leq x + 2 \leq 5$
$\underline{-2 \qquad -2 \quad -2}$

⇨ $-7 \leq x \quad \leq 3$

Notice that this double inequality means the same thing as:

$x + 2 \geq -5 \; and \; x + 2 \leq 5$

From here, it is easy to get the graph and the interval notation. We will use this simpler method whenever we have the absolute value "and" type problem from this point forward.

Note: This method should not be used for the absolute value "or" situation because the original double inequality ends up being a false statement. Consider Example 1 above: $|x+2| \geq 5$
We cannot rewrite this as the double inequality $-5 \geq x+2 \geq 5$ because this would imply that $-5 \geq 5$, and this is simply not true. You cannot begin your problem with a false statement and expect to end up with a true statement for an answer, right? This is why we do not use the double inequality for the absolute value "or" problems.

3. $|-2x+1| < 3$

$$|-2x+1| < 3$$

$$\Rightarrow \quad \begin{array}{rcl} -3 < & -2x+1 & < 3 \\ \underline{-1} & \underline{-1} & \underline{-1} \end{array}$$

$$\Rightarrow \quad -4 < -2x \quad < 2$$

$$\Rightarrow \quad \frac{-4}{-2} > \frac{-2x}{-2} \quad > \frac{2}{-2}$$

We must reverse both inequalities when dividing by -2.

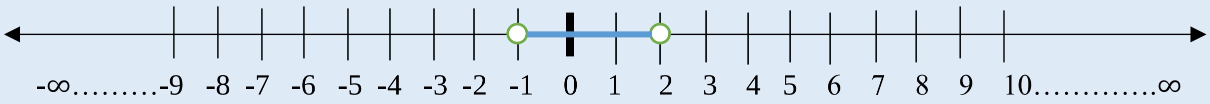
$$\Rightarrow \quad 2 > x > -1$$

Now, turn this around to read it from smaller to bigger:

$$-1 < x < 2$$

Reminder: The inequality symbols should stay pointing toward the same thing when you turn it around (and opening toward the same thing as well).

Now graph this on a number line:



Interval notation: $(-1,2)$

Sometimes, we will need to isolate the absolute value before we can rewrite it as a compound inequality (or split it into two inequalities) just as we did with absolute value equations. We won't even know if we have an "and" or an "or" situation until we isolate the absolute value because sometimes we will multiply or divide by a negative number in the process reversing the inequality. The next two examples illustrate isolation of the absolute value.

4. $2|x-3|+5 \geq 11$

$$\begin{array}{rll} & 2|x-3|+5 & \geq 11 \\ & \quad\;\; -5 & -5 \\ \hline ⇨ & 2|x-3| & \geq 6 \\ ⇨ & \frac{2|x-3|}{2} & \geq \frac{6}{2} \\ ⇨ & |x-3| & \geq 3 \end{array}$$

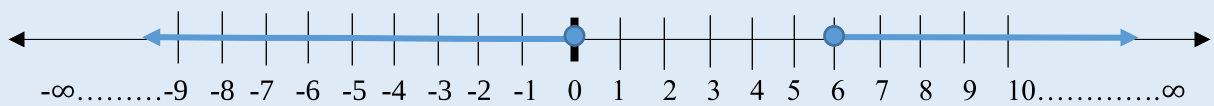
Now we can spit this into two inequalities with an “or” between them:

$$x - 3 \geq 3 \qquad \text{or} \qquad x - 3 \leq -3$$
$$\underline{\;+3 \quad + 3\;} \qquad\qquad \underline{\;+3 \quad + 3\;}$$

⇨ $x \geq 6$ ⇨ $x \leq 0$

So we have the solution “ $x \geq 6$ or $x \leq 0$ ”.

The graph is as follows:



Interval notation: $(-\infty, 0] \cup [6, \infty)$

5. $-3|4x - 2| + 7 < -11$

$$-3|4x - 2| + 7 < -11$$
$$\underline{\qquad\qquad -7 \quad - 7}$$

⇨ $-3|4x - 2| \quad < -18$

⇨ $\dfrac{-3|4x-2|}{-3} \quad > \dfrac{-18}{-3}$

⇨ $|4x - 2| \quad > 6$

Now we can spit this into two inequalities with an “or” between them:

$$\begin{array}{r} 4x - 2 > 6 \\ \underline{\quad +2 \;\; +2} \end{array} \qquad \text{or} \qquad \begin{array}{r} 4x - 2 < -6 \\ \underline{\quad +2 \;\; +2} \end{array}$$

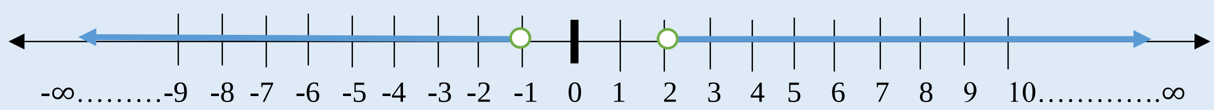
⇨ $4x > 8$ ⇨ $4x < -4$

⇨ $\frac{4x}{4} > \frac{8}{4}$ ⇨ $\frac{4x}{4} < \frac{-4}{4}$

⇨ $x > 2$ ⇨ $x < -1$

So we have the solution “ $x > 2$ or $x < -1$ ”.

The graph is as follows:



Interval notation: $(-\infty, -1) \cup (2, \infty)$

There are some absolute value problems that you can solve using logic, just like you did with some of the absolute value equations. You can still solve them out the long way, but with a little bit of thought before each problem, you can save a lot of time….

6. $|3x+1| > -10$

We know that the absolute value of any number is positive and every positive number is greater any negative number:

$$|anything| > negative\ number$$

Therefore the solution to this problem is "all real numbers" or $(-\infty, \infty)$.

We could also work this out to arrive at the same conclusion:

$$\begin{array}{r} 3x+1 > -10 \\ -1 \quad\ -1 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 3x+1 < 10 \\ -1 \quad -1 \\ \hline \end{array}$$

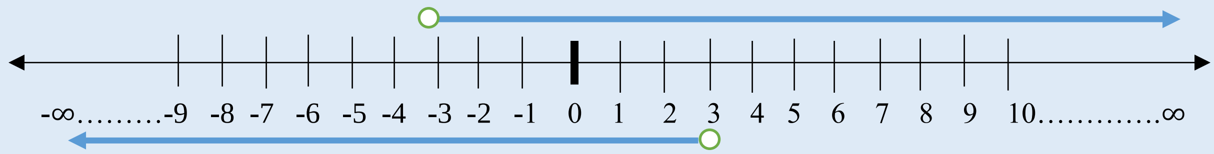
⇨ $3x > -11$ ⇨ $3x < 9$

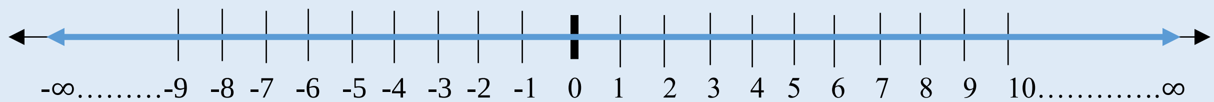
⇨ $\frac{3x}{3} > -\frac{11}{3}$ ⇨ $\frac{3x}{3} < \frac{9}{3}$

⇨ $x > -\frac{11}{3}$ ⇨ $x < 3$

So we have the solution " $x > -\frac{11}{3}$ $\left(-3\frac{1}{3}\right)$ or $x < 3$ ".

Graphing both of these on the same number line and taking "everything that gets covered" gives us the entire real line:





Interval notation: $(-\infty, \infty)$

It is much quicker to use logic for this one. It is good to check each problem before you begin to see if logic such as this applies.

7. $|3x + 1| < -10$

This example can also be approached using logic. Since any positive number cannot be smaller than a negative number, we know that this problem has no solution. We could also work this one out and we will get the same result:

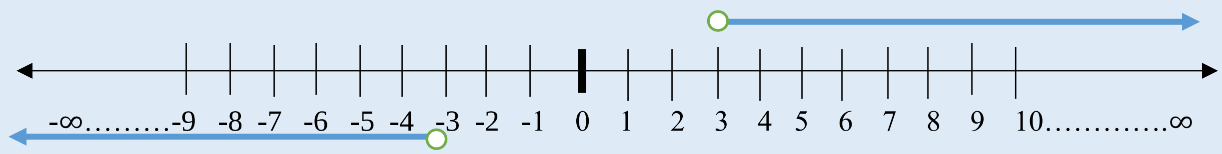
$$\begin{array}{r} 3x + 1 < -10 \\ \underline{\quad -1 \quad\; -1} \end{array} \qquad \text{and} \qquad \begin{array}{r} 3x + 1 > 10 \\ \underline{\quad -1 \quad -1} \end{array}$$

⇨ $3x < -11$ ⇨ $3x > 9$

⇨ $x < -\frac{11}{3}$ ⇨ $x > 3$

So we have the solution " $x < -\frac{11}{3}\left(-3\frac{1}{3}\right)$ and $x > 3$ ".

Graphing them both on the same number line, we get:



Now, take the intersection and you will see that there is none.

Therefore, there is no solution: $\emptyset$

8. $|3x + 1| \leq 0$

For this problem, we know the absolute value cannot be less than 0 (since that means it would be negative), so all we need to do is to set it equal to 0:

$$3x + 1 = 0$$

$$\Rightarrow \quad x = -\frac{1}{3}$$

The solution to this inequality is just a point, not an interval.