4.3 Solving Compound Linear Inequalities

The words that we use can be very powerful and in fact, even the small words, like “and” and “or” can change the entire meaning of a phrase significantly when they are interchanged. Before we apply this to mathematics, you can see it clearly in everyday language:

“Would you like to go to the beach or to the movies?”

“Would you like to go to the beach and to the movies?”

The only word that was changed in those two question was “or” to “and”, but how would your response change? For the first question, you might answer “the beach”. For the second question, you might answer “yes”. They mean something different.

Here is another example:

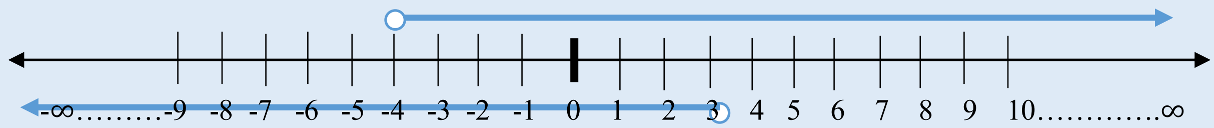
“It is sunny and it is raining.”

“It is sunny or it is raining.”

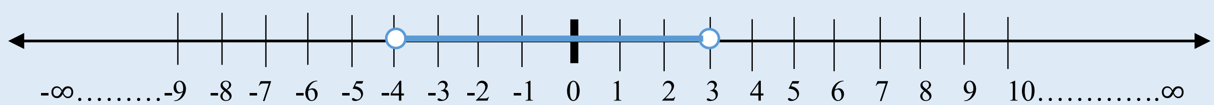
Once again, the only word that has changed is the connecting word “and” to “or”, but for the first statement to be true, both things have to be true at the same time. It has to be sunny *and* it also has to be raining. But for the second statement to be true, only one of them needs to be true. If it is sunny, but not raining, the second statement is still true. These statements mean something different.

These words are actually logical connectors and when applied to statements, they form *compound statements*. In mathematics, when we use them to join inequalities, we get *compound inequalities*. Now, let’s apply this to mathematics. The thing to keep in mind is that “and” means both have to be true and “or” means at least one of them is true. For this reason, when we apply this to inequalities, the “and” means to take the intersection or “overlap” since that is where both inequalities will be true.

$$x > -4 \text{ and } x < 3$$

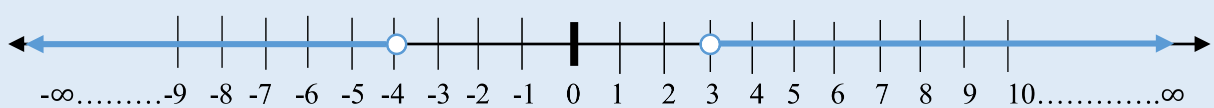


Where are both inequalities true? What values are both greater than -4 and smaller than 3 at the same time? Visually, we see it as the "overlap" of the solutions – where there are two blue lines instead of just one. Taking this "overlap" or intersection, we arrive at the following final graph:



Now, let's turn our attention to the word "or". If we have two inequalities with an "or" between them, then the statement will be true if either inequality is true. This means that any value that satisfies either inequality should be a part of the solution set, since that value will make the statement true. For this reason, we take all solutions of either inequality to be a part of the solution of the compound inequality. Looking at the graph, you can think of it as taking everything the graph covers:

$$x < -4 \text{ or } x > 3$$



So we keep both pieces for our final graph in this example.

Before we embark further on inequalities, there are some symbols we need to know:

$\cap$ This symbol means "intersect" and it corresponds to "and"

$\cup$ This symbol means "union" and it corresponds to "or"

To better understand the meaning of these symbols, we will consider their use with sets in the examples below:

<u>Examples</u>

Given the following sets, perform the indicated operations:

$A = \{0,1,2,3,4,5,6\}$

$B = \{4,6,8,10\}$

$C = \{-3,-1,0,1,2,3\}$

$D = \{-3,1,2,3,5,8\}$

1. $A \cap B$

   The symbol $\cap$ means intersection, so we should look for what A and B have in common.

$$A \cap B = \{4,6\}$$

2. $B \cap C$

   We are again taking the intersection, so look for what B and C have in common. In this case, these two sets have nothing in common, so the answer is the empty set:

$$B \cap C = \emptyset$$

3. $A \cup B$

The symbol $\cup$ means union, so we should take everything that A and B have and put it together on one set.

$$A \cup B = \{0,1,2,3,4,5,6,8,10\}$$

4. $C \cup D = \{-3, -1,0,1,2,3,5,8\}$

Do the same thing here for the sets C and D.

$$C \cup D = \{-3, -1,0,1,2,3,5,8\}$$

5. $C \cap D$

Take the intersection of the sets C and D:

$$C \cap D = \{-3,1,2\}$$

The examples that we just did are called "discrete" examples because we are working on sets with members that can be counted rather than on intervals. We did these first because they help us to get a more concrete idea of what the union and the intersection really mean. We will now apply these concepts to intervals which happen to be solutions of linear inequalities.

<u>Examples</u>

Solve each of the following compound inequalities, graph the solution set on a number line, and write the solution set in interval notation.

1. $5(x+1) \le 4(x+3)$ *and* $x + 12 > -3$

First, we will solve each inequality:

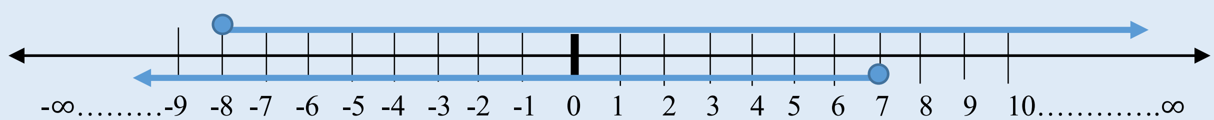
$5(x+1) \le 4(x+3)$ and $x + 5 > -3$

$$\begin{array}{rl} & 5(x+1) \le 4(x+3) \\ \Rightarrow & 5x + 5 \le 4x + 12 \\ & \underline{-4x \qquad\quad - 4x} \\ \Rightarrow & x + 5 \le 12 \\ & \underline{\quad -5 \quad -5} \\ \Rightarrow & x \le 7 \end{array}$$

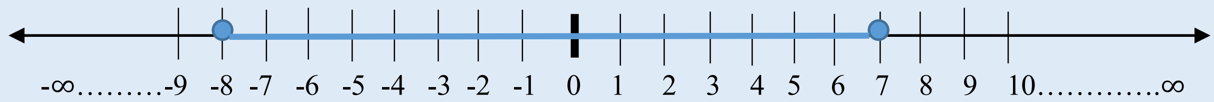
$$\begin{array}{rl} & x + 5 > -3 \\ & \underline{\quad -5 \quad -5} \\ \Rightarrow & x > -8 \end{array}$$

So we have the solution " $x \le 7$ and $x \ge -8$ ".

Before writing this in interval notation, we should graph both inequalities together on the same number line and take their intersection (since this what "and" means). This will make writing the interval notation very easy.



Now, if we take the intersection of the two graphs above (or the "overlap"), we get the following final graph:



Now, it is easy to read the graph from left to right to get the interval notation: $[-8,7]$.

Note that if we wrote the pieces separately in interval notation, which would look like this $(-\infty, 7] \cap [-8, \infty)$, we would not be writing it in the most simplified form. We need to actually take the intersection " $\cap$ " to finish the problem. (Leaving it in this form is like leaving your answer as $9 - 5$ instead of writing 4.)

2. $5(x+1) \leq 4(x+3)\ or\ x + 12 > -3$

Notice that the inequalities are the same as in example 1, but they are connected by "or" instead of "and". Solving the inequalities will be the same, but when we look at the graph, we will take the union of the two intervals instead of the intersection:

$5(x+1) \leq 4(x+3)$ or $x + 5 > -3$

$\underline{\quad -5 \quad -5}$

⇨ $5x + 5 \leq 4x + 12$ ⇨ $x > -8$

$\underline{-4x \qquad -4x}$

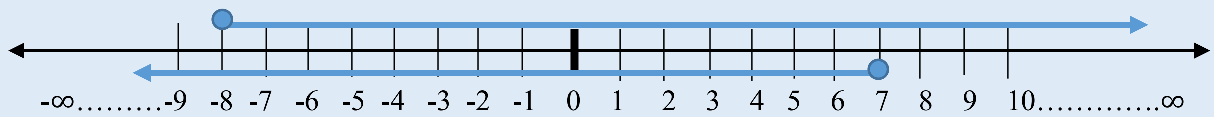
⇨ $x + 5 \leq 12$

$\underline{\quad -5 \quad -5}$

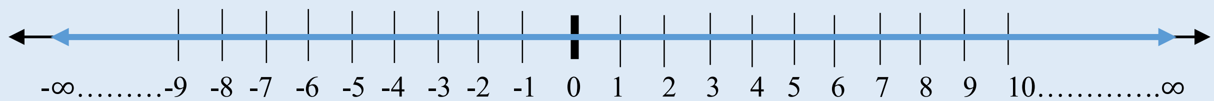
⇨ $x \leq 7$

So we have the solution " $x \leq 7$ or $x \geq -8$ ".

Now graph both inequalities together on the same number line and take their union (since this what "or" means). This will make writing the interval notation very easy.



Now, if we take the union of the two graphs above (or "everything covered"), we get the following final graph:



Now, it is easy to read the graph from left to right to get the interval notation: $(-\infty, \infty)$. Notice that everything on the number line gets covered by the union of these two intervals.

Once again, if we wrote the pieces separately in interval notation, which would look like this $(-\infty, 7] \cup [-8, \infty)$, we would not be writing it in the most simplified form. We need to actually take the union " $\cup$ " to finish the problem.

3. $4x < -12 \;\; or \;\; \frac{x}{2} > 4$

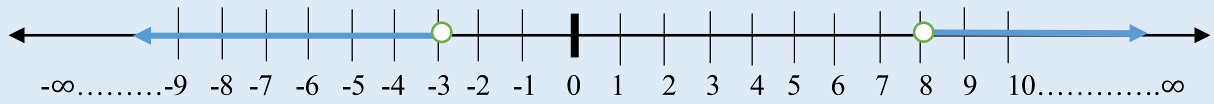
$4x < -12$ or $\frac{x}{2} > 4$

⇨ $\frac{4x}{4} < \frac{-12}{4}$ ⇨ $2 \cdot \frac{x}{2} > 4 \cdot 2$

⇨ $x < -3$ ⇨ $x > 8$

So we have the solution " $x < -3$ or $x > 8$ ".

Now graph both inequalities together on the same number line and take their union.



Now, it is easy to read the graph from left to right to get the interval notation: $(-\infty, 3) \cup (8, \infty)$. Here we have no choice but to write this union as two separate pieces since the pieces are "disjoint" or completely separated. We cannot simplify it any more.

If example 3 had an "and" instead of an "or", how would your answer be different? The process of solving the inequalities would be the same, but when you look at the two separate pieces on the graph and take the intersection, you would see that there is none. These pieces do not overlap. Therefore your answer would be "No solution" or the empty set $\emptyset$.

4. $5(x-2) \geq 0 \; and \; -3x < 9$

$5(x-2) \geq 0$ and $-3x < 9$

⇨ $5x - 10 \geq 0$ ⇨ $\frac{-3x}{-3} > \frac{9}{-3}$

$\quad +10 \quad +10$

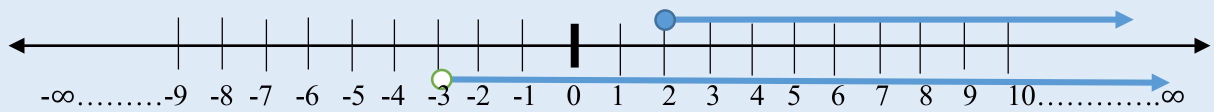
⇨ $5x \geq 10$ ⇨ $x > -3$

⇨ $\frac{5x}{5} \geq \frac{10}{5}$

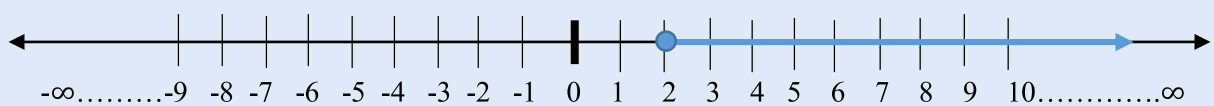
⇨ $x \geq 2$

So we have the solution " $x \geq 2$ and $x > -3$ ".

Now we will graph both inequalities together on the same number line and take their intersection (since this what "and" means). Notice that the directions are the same for these inequalities, but it doesn't matter – just find where they overlap on the graph:



Now, if we take the intersection of the two graphs above (or the "overlap"), we get the following final graph:



Interval notation: $[2, \infty)$.

5. $2x - 5 \leq 3x + 1 \; or \; x - 3 > 4$

$$\begin{array}{rl} 2x - 5 \leq 3x + 1 \\ \underline{-3x \qquad -3x} \end{array} \qquad \text{or} \qquad \begin{array}{r} x - 3 > 4 \\ \underline{+3 \; +3} \end{array}$$

⇨ $-x - 5 \leq 1$ ⇨ $x > 7$

$\underline{\quad +5 \; +5}$

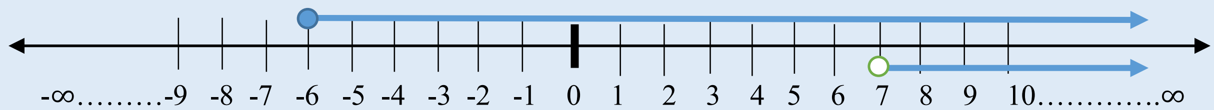
⇨ $-x \quad \leq 6$

⇨ $\dfrac{-x}{-1} \geq \dfrac{6}{-1}$

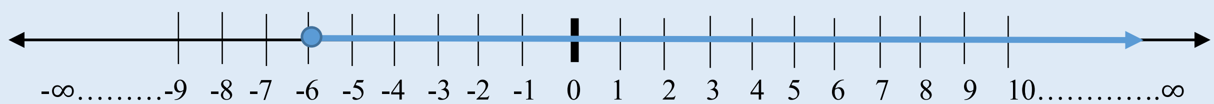
⇨ $x \geq -6$

So we have the solution " $x \geq -6$ or $x > 7$ ".

Now graph both inequalities together on the same number line and take their union (since this what "or" means).



Now, if we take the union of the two graphs above (or "everything covered"), we get the following final graph:



The interval notation for this graph is : $[-6, \infty)$.
Once again, if we wrote the pieces separately in interval notation, which would look like this $[-6, \infty) \cup (7, \infty)$, we would not be writing it in the most simplified form. We need to actually take the union " $\cup$ " to finish the problem, and the graph helps us to do this.

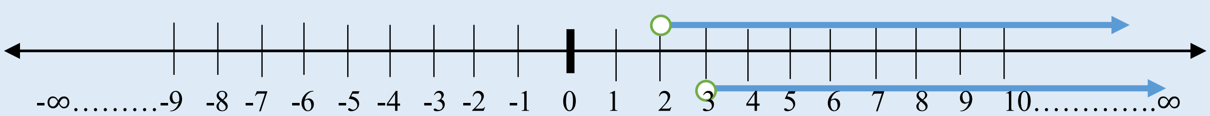
What if the example we just finished had been an "and" problem instead of an "or" problem? In that case, when we look at the two intervals on the first graph, we would have to take the overlap of the two, which would be the piece from 7 to infinity: $(7, \infty)$. That little word makes a big difference in the answer!

6. $2x > x + 3 \;\; and \; -\frac{x}{8} + 1 < \frac{3}{4}$

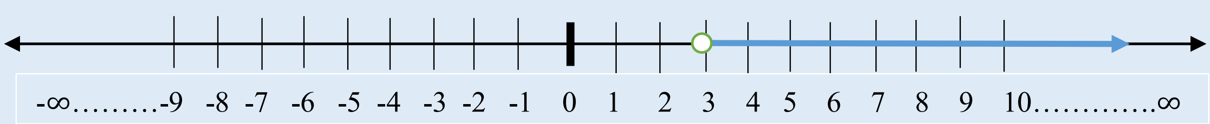
$$\begin{array}{rl} & 2x > x + 3 \\ & \underline{-x \quad -x} \\ \Rightarrow & x > 3 \end{array} \qquad \text{and} \qquad \begin{array}{rl} & -\frac{x}{8} + 1 < \frac{3}{4} \\ \Rightarrow & -8\left(-\frac{x}{8} + 1\right) > \frac{3}{4}(-8) \\ \Rightarrow & x - 8 > -6 \\ & \underline{\;+8 \quad +8\;} \\ \Rightarrow & x > 2 \end{array}$$

So we have the solution " $x > 3$ and $x > 2$ ".

Graphing both inequalities together on the same number line:



Now, if we take the intersection of the two graphs above (or the "overlap"), we get the following final graph:



The interval notation for this graph is: $(3, \infty)$.

The last two examples involve double sided inequalities, which we have not really seen except in the first chart introducing interval notation.

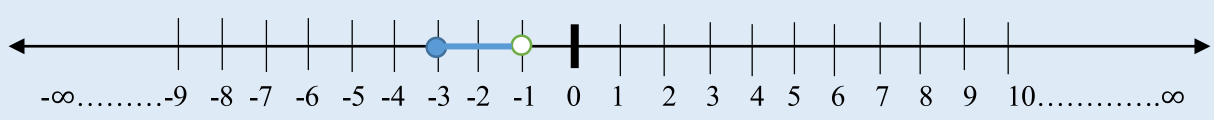
Double sided inequalities represent two inequalities with an “and” between them. For example, the double sided inequality $-3 < x < 2$ means the same thing as the compound inequality "$x > -3 \; and \; x < 2$". It is much easier to work with them in the double sided form, as you will see.

7. $-1 \leq 2x + 5 < 3$

For a double-sided inequality, we can work on “all three sides” at the same time in order to isolate the variable in the middle. First, subtract 5 from all three sides, then divide all of them by 2:

$$\begin{array}{rl} & -1 \leq 2x + 5 < 3 \\ & -5 \qquad\; -5 \quad -5 \\ \hline \Rightarrow & -6 \leq 2x \qquad < -2 \\ \Rightarrow & \frac{-6}{2} \leq \frac{2x}{2} < \frac{-2}{2} \\ \Rightarrow & -3 \leq x < -1 \end{array}$$

This means that that the solution consists of all of the real numbers between -3 and 1 including the endpoint -3. This form makes it very easy to graph since you can see that x is “trapped” between -3 and 1.



The interval notation is also very easy to read from this form. You can see that we need a bracket on -3 and a parentheses on -1 from looking at the symbols next to each of them in the inequality. The interval notation is as follows: $[-3, -1)$

We could also have viewed this problem as a compound inequality and approached it as follows:

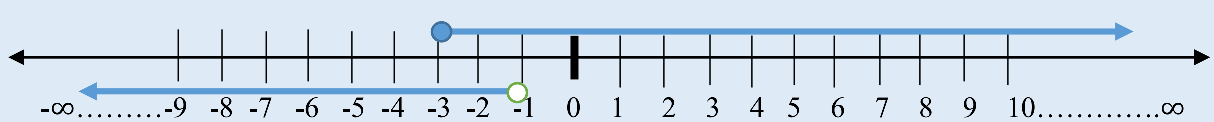
$-1 \leq 2x + 5 < 3$ means the same thing as the compound inequality "$2x + 5 \geq -1$ *and* $2x + 5 < 3$"

Solve each inequality:

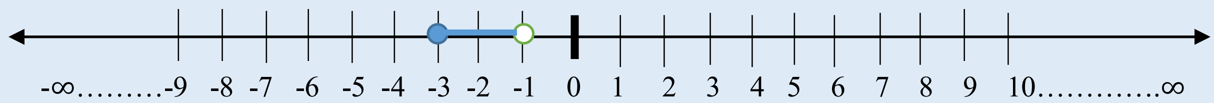
$$2x + 5 \geq -1 \qquad \text{and} \qquad 2x + 5 < 3$$

⇨ $\begin{array}{r} 2x + 5 \geq -1 \\ -5 \quad -5 \\ \hline \end{array}$	⇨ $\begin{array}{r} 2x + 5 < 3 \\ -5 \; -5 \\ \hline \end{array}$
⇨ $2x \geq -6$	⇨ $2x < -2$
⇨ $\frac{2x}{2} \geq \frac{-6}{2}$	⇨ $\frac{2x}{2} < \frac{-2}{2}$
⇨ $x \geq -3$	⇨ $x < -1$

So we have the solution " $x \geq -3$ and $x < -1$ ".



Now, if we take the intersection of the two graphs above (or the "overlap"), we get the following final graph:



Interval notation: $[-3,1)$.

You can see that both methods will yield the same answer, but the first method is much more efficient for double sided inequalities.

8. $-4 < -2x + 7 \ \leq 12$

Using the first method above for double-sided inequalities:

$$\begin{array}{rl} & -4 < -2x + 7 \ \leq 12 \\ & \underline{-7 \qquad\quad -7 \quad -7} \\ \Rightarrow & -11 < -2x \qquad \leq 5 \end{array}$$

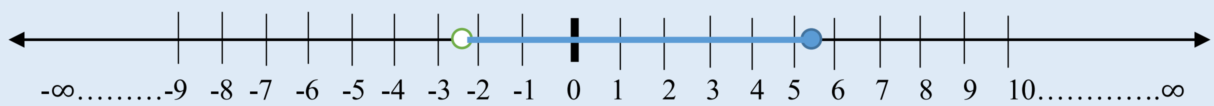
$$\Rightarrow \quad \frac{-11}{-2} < \frac{-2x}{-2} \leq \frac{5}{-2}$$

$$\Rightarrow \quad \frac{11}{2} > \ x \ \geq -\frac{5}{2}$$

Prior to graphing this, you should turn it around so that you can read it from "smaller to bigger". You should always have "less than" rather than "greater than" symbols if it is in order from smallest to greatest….The symbols will still point to the same thing and be open toward the same things. That will not change:

⇨ $$-\frac{5}{2} \leq x < \frac{11}{2}$$

Note that $-\frac{5}{2} = -2\frac{1}{2}$ and $\frac{11}{2} = 5\frac{1}{2}$ for ease in graphing.



The interval notation is as follows: $\left[-\frac{5}{2}, \frac{11}{2}\right)$