4.2 Solving Linear Inequalities

An *inequality* looks just like an equation with the " = " sign replaced by one of the following symbols: $<, \leq, >, \geq$

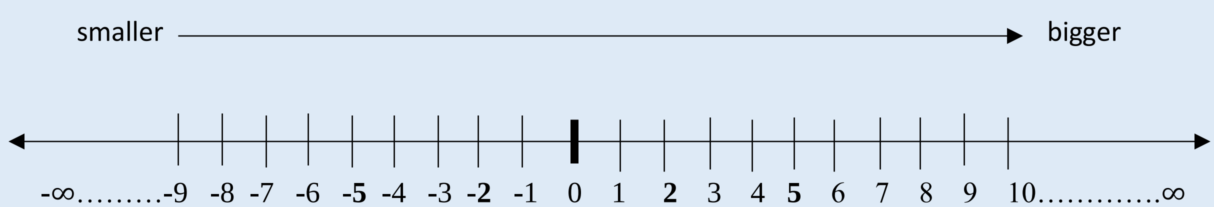
Each of these symbols is a comparison symbol between the two sides. The " = " sign tells us that the two sides are equal, whereas these signs tell us that one side is either bigger or smaller than the other. Equality is also allowed if the bar is included beneath the symbol. The following is a chart that summarizes the meaning of each symbol:

Symbol	**Meaning**	**Examples**	**In words**
$<$	*is* less than	$5 < 7$	5 is less than 7
$\leq$	*is* less than or equal to	$5 \leq 7$ $7 \leq 7$	5 is less than or equal to 7 (5 is less than 7, so this is true) 7 is less than or equal to 7 (7 is equal to 7, so this is true)
$>$	*is* greater than	$7 > 5$	7 is greater than 5
$\geq$	*is* greater than or equal to	$7 \geq 5$ $7 \geq 7$	7 is greater than or equal to 5 (7 is greater than 5, so this is true) 7 is greater than or equal to 7 (7 is equal to 7, so this is true)

The word "*is*" is important in the meanings above. This word indicates comparison. When we are translating words into math, it matters whether you see the word "*is*" there or not. If you just see the words *less* *than*, for instance, you should be thinking about subtraction. For example, the words "4 less than a number" translate to x $-$ 4. However, the words "4 *is* less than a number" translate to $4 < x$.

Notice that the same thing can be said in more than one way, just as with equations. If you say $x = 4$, that is the same thing as saying $4 = x$. You just turn the equation around. A similar thing is true for inequalities. If you say $x < 4$ (x is less than 4), that is the same $4 > x$ (4 is greater than x). So in our chart above, $5 \leq 7$ and $7 \geq 5$ actually just say the same thing. Notice that the symbol always opens toward the larger one and points toward the smaller one. If you turn an inequality around, you must turn the symbol around as well (reverse its direction). Sometimes it will be convenient to do this to our solutions when we are graphing our solutions on a number line.

But, how do we solve linear inequalities? We can actually apply the same technique that we use for linear equations, with one additional rule: Whenever you multiply or divide by a negative number, you must reverse your inequality. The reason for this stems from the order of the numbers on the number line. If you look at the negatives, they are in the opposite order moving away from 0. For example, we know that $2 < 5$, but now compare -2 and -5. Realizing that the smaller numbers are always on the left and the larger numbers are always on the right, you must admit that $-5 < -2$, or you can say $-2 > -5$.



Considering these two inequalities together, you can see that it makes sense that when we multiply or divide by a negative number, we must reverse the inequality to keep the statement true:

$$2 < 5$$

$$-2 > -5$$

Dividing both sides by -1 reverses the inequality.

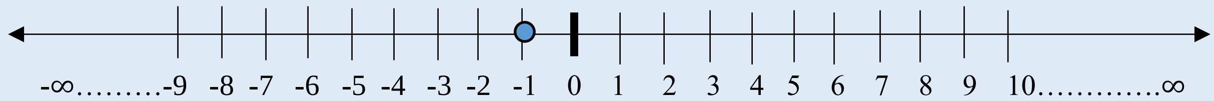
Let's consider an equation and an inequality side by side so that we can observe the similarities and differences between them:

Equation	Inequality
$-3x + 2 = 5$	$-3x + 2 \leq 5$
$\quad -2 \quad -2$	$\quad -2 \quad -2$
$-3x \quad = 3$	$-3x \quad \leq 3$
⇨ $\dfrac{-3x}{-3} = \dfrac{3}{-3}$	⇨ $\dfrac{-3x}{-3} \geq \dfrac{3}{-3}$
⇨ $x = -1$	⇨ $x \geq -1$

As soon as we divide by -3, we must reverse the inequality. (If we were dividing by a positive number, we would keep the inequality the same).

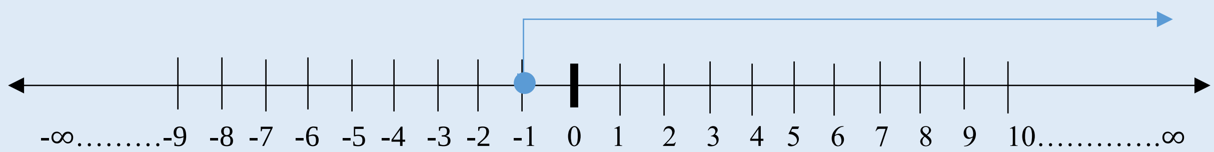
If we were to represent these solutions on a number line, however, they would look quite different. This is because they mean something quite different from each other. The equation gives us one solution, $x = -1$, and that is simply represented by a point on the number line at -1. The inequality gives us infinitely many solutions since $x \geq -1$ means any number greater than or equal to -1. In short, any number to the right of -1 will solve the inequality $-3x + 2 \leq 5$. Try plugging in any number greater than -1 and you will see that it is true….

For the solution to an equation, we have a point:



For the inequality, we do not simply get a point on the number line, but rather we get an *interval*. An interval is essentially a piece of the number line containing infinitely many points.

For the solution to an inequality, we have an interval:



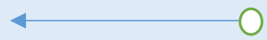
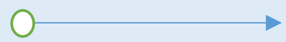
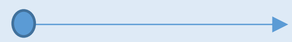
We could write this interval in the old familiar set notation as $\{x|x \geq -1\}$ (which reads " The set of all x such that x is greater than or equal to -1").

This interval could also be written in *interval notation* as $[-1, \infty)$ which reads "The interval from -1 to ∞ including the endpoint -1".

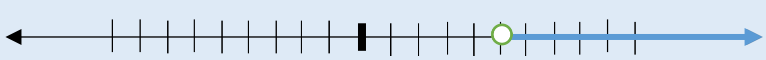
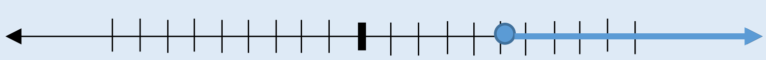
Interval notation is our preferred notation in this course for all inequality solutions because it is simpler to read and understand, especially when we end up with multiple intervals as a part of our solution (in future sections). It will also help us describe the domain and range of a function when we get to the graphing sections.

To write a solution to an inequality using interval notation it is best to just read the graph from left to right to get the numbers inside. Where does the interval begin and where does it end? To get the enclosing symbols (brackets vs. parentheses), just decide whether or not the endpoint is included. If it is included, use a bracket. If it is not, use parentheses. Note that both $-\infty$ and ∞ will always have a parentheses because we cannot ever get there….

Below is a summary of interval notation:

Symbol	**Endpoint on graph**	**Direction on graph**	**Interval notation**
$<$	Open hole or parentheses		parentheses
$\leq$	Closed hole or bracket		bracket
$>$	Open hole or parentheses		parentheses
$\geq$	Closed hole or bracket		bracket

Some examples of intervals with their graphs and interval notation:

Inequality	**Graph**	**Interval Notation**
$x > 5$		$(5, \infty)$
$x \geq 5$		$[5, \infty)$
$x < 3$		$(-\infty, 3)$
$x \leq 3$		$(-\infty, 3]$
$-2 < x < 7$		$(-2, 7)$
$-2 \leq x < 7$		$[-2, 7)$

Examples

Solve each of the following inequalities, graph the solution set on a number line, and write the solution set in interval notation.

1. $-5t + 3 \leq 5$

First, we will solve the inequality:

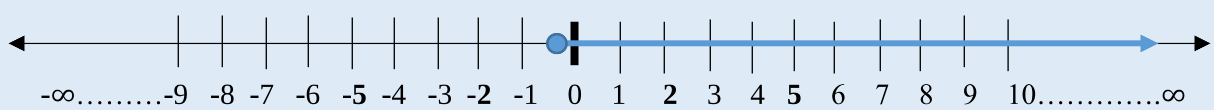
$$\begin{array}{rcl} -5t + 3 & \leq & 5 \\ -3 & & -3 \\ \hline -5t & \leq & 2 \end{array}$$

$$⇨ \quad \frac{-5t}{-5} \geq \frac{2}{-5}$$

When we divide by -5, we have to reverse the inequality.

$$⇨ \quad t \geq -\frac{2}{5}$$

Next, we will graph this inequality on a number line. That will make it easy to write down the interval notation.



The endpoint at $-\frac{2}{5}$ should have a closed hole (or a bracket if you prefer) since our symbol $\geq$ includes the endpoint and the graph should point in the greater than direction (to the right).

Note that $-\frac{2}{5}$ is between -1 and 0 (since $-\frac{5}{5} < -\frac{2}{5} < \frac{0}{5}$ and also it is closer to 0 than -1).

To write this in interval notation, just read the graph from left to right to see that it starts at $-\frac{2}{5}$ and goes toward ∞. There should be a bracket on $-\frac{2}{5}$ and a parentheses on ∞: $\left[-\frac{2}{5}, \infty\right)$

2. $3(z-2) \leq 2(z+7)$

First, we will solve the inequality:

$$3(z-2) \leq 2(z+7)$$

$$\Rightarrow \quad 3z - 6 \leq 2z + 14$$

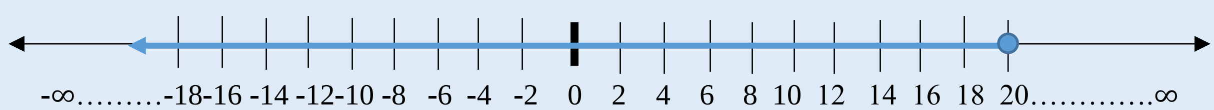
$$\underline{-2z \qquad -2z}$$

$$\Rightarrow \quad z - 6 \leq 14$$

$$\underline{+6 \qquad +6}$$

$$\Rightarrow \quad z \leq 20$$

Next, we will graph this inequality on a number line.



The endpoint at -20 should have a closed hole (or a bracket if you prefer) since our symbol $\leq$ includes the endpoint and the graph should point in the less than direction (to the left).

Notice that we scaled our graph by two's. You can scale your graph by any number you like.

To write this in interval notation, just read the graph from left to right to see that it starts from $-\infty$ and ends at 20. There should be a parentheses on $-\infty$ and a bracket on -20 : $(-\infty, 20]$

3. $\frac{6-y}{-2} < -6$

First, we will solve the inequality:

Instead of subtracting 6 to both sides, we could have added y and subtracted 12 to both sides, but we would need to turn the inequality around so that the variable is on the left to read it for graphing:

$$\begin{array}{rcl} 6 - y & > & 12 \\ -12 & & -12 \\ \hline -6 - y & > & 0 \\ +y & & +y \\ \hline -6 & > & y \end{array}$$

But $-6 > y$ is misleading when we graph, so rewrite it as $y < -6$. Now it is clear which direction the arrow should go.

$$\frac{6-y}{-2} < -6$$

When we multiply by -2, we have to reverse the inequality.

⇨ $$-2\left(\frac{6-y}{-2}\right) > -6(-2)$$

⇨ $$\begin{array}{rcl} 6 - y & > & 12 \\ -6 & & -6 \\ \hline \end{array}$$

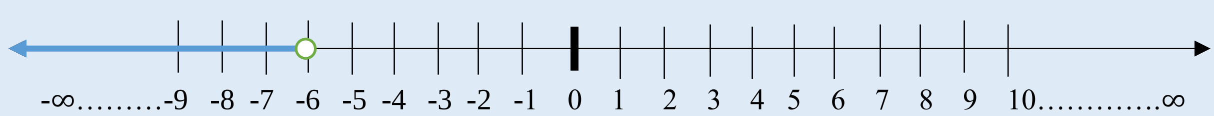
⇨ $$-y > 6$$

We need to divide (or multiply) by -1 now in order to isolate y. This means we have to reverse the inequality again.

⇨ $$\frac{-y}{-1} < \frac{6}{-1}$$

⇨ $$y < -6$$

Next, we will graph this inequality on a number line.



The endpoint at -6 should have an open hole (or a parentheses if you prefer) since our symbol $<$ does not include the endpoint and the graph should point in the less than direction (to the left).

To write this in interval notation, just read the graph from left to right to see that it starts from $-\infty$ and ends at -6. There should be a parentheses on $-\infty$ and on -6 : $(-\infty, -6)$

4. $\frac{x-2}{2} - \frac{x-1}{5} \geq -\frac{x}{4}$

First, we will solve the inequality by finding the common denominator and multiplying both sides by it:

$$\frac{x-2}{2} - \frac{x-1}{5} \geq -\frac{x}{4}$$

$$\Rightarrow \quad 20\left(\frac{x-2}{2} - \frac{x-1}{5}\right) \geq \left(-\frac{x}{4}\right)20$$

$$\Rightarrow \quad 10(x-2) - 4(x-1) \geq -5x$$

$$\Rightarrow \quad 10x - 20 - 4x + 4 \geq -5x$$

$$\Rightarrow \quad 6x - 16 \geq -5x$$

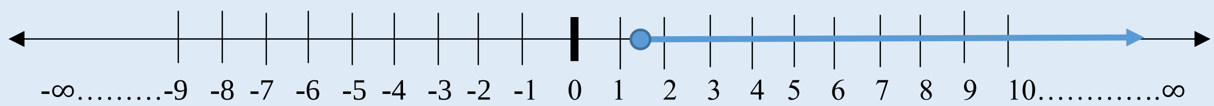
$$\Rightarrow \quad \begin{array}{r} 6x - 16 \geq -5x \\ \underline{+5x \qquad\quad +5x} \\ 11x - 16 \geq 0 \end{array}$$

$$\Rightarrow \quad 11x \geq 16$$

$$\Rightarrow \quad x \geq \frac{16}{11} \; or \; 1\frac{5}{11}$$

Note: You can ONLY multiply both sides by the LCD when it is a number, not a variable. If there is a variable in the denominator, then you don't know if you are multiplying by a positive or a negative, so you don't know if you need to reverse the inequality. The same is true with cross multiplying for the same reason. In these cases, you will just need to work with (add/subtract) the fractions.

It is helpful to rewrite any fractional answers as mixed numbers (or decimals) in order to find them on the graph.



The interval notation is: $\left[\frac{16}{11}, \infty\right)$

Sometimes strange things happen when we are solving, as in the next two examples:

5. $\frac{3b+7}{3} \leq \frac{2b-9}{2}$

$$\frac{3b+7}{3} \leq \frac{2b-9}{2}$$

$$\Rightarrow \quad 2(3b+7) \leq 3(2b-9)$$

$$\Rightarrow \quad 6b + 14 \leq 6b - 27$$

$$\Rightarrow \quad \begin{array}{r} 6b + 14 \leq 6b - 27 \\ \underline{-6b \qquad\quad - 6b} \end{array}$$

$$\Rightarrow \quad 14 \leq -27$$

Notice that the variable dropped out of the equation! When this happens, we must evaluate the remaining inequality to be either true or false. If it is true, then any real number should solve the inequality. If it is false, then there is no real number that will solve the inequality.

Since 14 is NOT less than or equal to -27, this inequality has no solution (often called the empty set denoted by the symbol 0.

6. $-1+4(y-1)+2y \leq \frac{1}{2}(12y-30)+15$

$$\Rightarrow \quad -1+4y-4+2y \leq 6y-15+15$$

$$\Rightarrow \quad 6y-5 \leq 6y$$

$$\underline{-6y \qquad -6y}$$

$$\Rightarrow \quad -5 \leq 0$$

Once again, the variable dropped out, but this time we end up with a true statement. This means that any number you plug in for y will make the original inequality true. So the answer is "all real numbers", or in interval notation: $(-\infty, \infty)$.

Applications

It is often useful to use inequalities to solve problems where limitations exist, often financial or physical. For example, if you own a business, you will have limitations on the number of people you can hire, the number of hours they will work, the type of building you can afford, etc. There are often physical limitations on equipment such as how much weight a car or an airplane can hold or how much of a certain chemical can be tolerated in the water supply, etc. We can use inequalities to represent problems involving these kinds of quantities in much the same way we did with equations. We will consider a couple of examples of this nature.

Examples

Set up an inequality to represent the given problem and solve it.

1. Your business plans to open another store this coming year, but needs to make a profit of at least $35,750 in order to be able to afford the start-up costs. If your profit model is based on the number of units sold and is given by $P = .07x$ (basically you profit 7 cents for every item you sell), then how many units need to be sold this year in order to open up that new store?

   Solution: You need your profit to be greater than or equal to $35,750 since this is what "at least" means. Therefore, you can set up the inequality:

$$P \geq 35{,}750.$$

   But we need our inequality to be in terms of the variable x, since x represents the number of units sold and that is what we are looking for. Replace P with $.07x$ in the inequality:

$$.07x \geq 35{,}750.$$

   Now, just solve for x by dividing both sides by $.07$:

$$\frac{.07x}{.07} \geq \frac{35{,}750}{.07}$$

$$x \geq 510{,}714.286$$

   So, you have to sell $510{,}715$ units (we had to round up to the next whole number since you cannot sell .286 of a unit).

2. You are looking for a family vehicle that will accommodate large family trips and you need to make sure it will carry all 8 members of your family as well as luggage. You have two vehicles in mind, but your favorite holds no more than 1580 pounds. You are using an average weight of 145 pounds per person to make your calculations, so how much luggage would this vehicle allow each person to bring?

Solution: Total weight needs to be less than or equal to 1580. This total weight is a sum of passengers and luggage.

$$Weight \leq 1580$$

$$Passengers + Luggage \leq 1580$$

$$145(8) + L \leq 1580$$

$$1160 + L \leq 1580$$

$$\underline{-1160 \qquad -1160}$$

$$L \leq 420$$

So the total allowance for luggage is 420 pounds. To find out how much each person can bring, we can divide this by 8 to get 52.5 pounds each.