3.4 Logarithmic Equations

First, we will review simple logarithmic equations like the ones we did in section 3.1. These logarithmic equations are simple because all we really need to do is to rewrite our logarithm as an exponential in order to solve it. For more complicated logarithmic equations, such as equations involving sums and differences of logarithms, we will use the properties from section 3.2 to condense our logarithms so that we can rewrite our logarithms as exponentials.

Examples

To solve logarithmic equations, rewrite the logarithmic equation as an exponential equation using the definition of a logarithm. When there is only one logarithm, this is easy. Let's review with a few examples before considering equations with multiple logarithms.

1. $\log_2 x = -3$ — Rewrite in exponential form. (Slide the base underneath the other side.)

   $x = 2^{-3}$ — Use reciprocal property to eliminate the negative exponent.

   $x = \frac{1}{2^3}$ — Simplify.

   $x = \frac{1}{8}$

2. $\ln 4x = 3$ — Rewrite in exponential form.

   $\log_e x = 3$ — Recall: $\ln x = \log_e x$, so use base e.

   $4x = e^3$ — Divide by 4 to isolate x.

$$\frac{4x}{4} = \frac{e^3}{4}$$ Simplify.

$$x = \frac{e^3}{4} \approx 5.0214$$

Exact answer   Approximation using calculator

3. $\log_{16} x = \frac{1}{2}$ Rewrite in exponential form.

$x = 16^{\frac{1}{2}}$ Write as a radical (square root).

$x = \sqrt{16}$ Simplify.

$x = 4$

Recall that the "x" in the expression, $\log_b x$, is called the argument and that the argument of a logarithm must be positive $(x > 0)$ in order for the logarithm to be defined as a real number. Thus we must check every solution to a logarithmic equation to make sure that the argument is positive. Now we can consider examples which have more than one logarithm. Our technique is to get all of our logarithms on one side and rewrite them as a single logarithm. We can use the properties of logarithms to do this. Once we have a single logarithm, we are back to the same old thing!

4. $\log x - \log(x+3) = 1$ Rewrite as a single log.

$\log \frac{x}{x+3} = 1$ Rewrite in exponential form.

$\log_{10} \frac{x}{x+3} = 1$ (Recall: $\log x = \log_{10} x$)

$$\frac{x}{x+3} = 10^1 \quad \text{Clear fraction.}$$

$$\frac{\cancel{x+3}}{1} \cdot \frac{x}{\cancel{x+3}} = 10(x+3) \quad \text{Distribute 10 and simplify.}$$

$$\begin{array}{rl} x = & 10x + 30 \\ -10x & -10x \\ \hline -9x = & 30 \end{array} \quad \begin{array}{l} \text{Solve for } x. \\ \\ \text{Divide by } -9. \end{array}$$

$$\frac{-9x}{-9} = \frac{30}{-9} \quad \text{Simplify.}$$

$$x = -\frac{10}{3}$$

Since this solution makes the argument negative, there is no solution!

To solve a log equation with multiple log terms, get all the logs on the same side and then rewrite as a single log.

5. $\log_4(x+3) = 2 + \log_4(x-5)$ Get logs on one side first so that they can be combined.

$$\begin{array}{rl} \log_4(x+3) & = 2 + \log_4(x-5) \\ -\log_4(x-5) & \quad -\log_4(x-5) \\ \hline \log_4(x+3) - \log_4(x-5) & = 2 \end{array}$$

$$\log_4 \frac{x+3}{x-5} = 2 \quad \text{Rewrite in exponential form.}$$

$$\frac{x+3}{x-5} = 4^2 \quad \text{Simplify.}$$

$$\frac{x+3}{x-5} = 16 \quad \text{Clear fractions.}$$

$$\frac{\cancel{x-5}}{1} \cdot \frac{x+3}{\cancel{x-5}} = 16(x-5) \qquad \text{Distribute.}$$

$$x + 3 = 16x - 80 \qquad \text{Solve.}$$

$$\begin{array}{rcl} x+3 &=& 16x-80 \\ -x & & -x \\ \hline 3 &=& 15x - 80 \\ +80 & & +80 \\ \hline 83 &=& 15x \end{array}$$

$$\frac{83}{15} = \frac{\cancel{15}x}{\cancel{15}} \qquad \text{Simplify.}$$

$$x = \frac{83}{15}$$

Since $\frac{83}{15} = 5\frac{8}{15}$ which is greater than 5, the arguments are both positive and this is a valid solution to the log equation.

6. $\log_2 x + \log_2 8x - 3 = 2$ Isolate the log terms on one side.

$$\begin{array}{rcl} \log_2 x + \log_2 8x - 3 &=& 2 \\ +3 & & +3 \\ \hline \log_2 x + \log_2 8x &=& 5 \end{array}$$

$\log_2 x + \log_2 8x = 5$ Rewrite as a single log.

$\log_2(x \cdot 8x) = 5$ Simplify.

$\log_2(8x^2) = 5$ Rewrite in exponential form.

$8x^2 = 2^5$ Simplify.

$8x^2 = 32$ Continue isolating x.

$$\frac{8x^2}{8} = \frac{32}{8}$$

$x^2 = 4$ Solve by taking the square root of both sides, then simplify.

$\sqrt{x^2} = \pm\sqrt{4}$

$x = \pm 2$ Note that -2 will make the argument negative, so it is extraneous.

$x = 2$

7. $2\log_2 x = 3 + \log_2(x-2)$

$$\begin{array}{rr} 2\log_2 x & = 3 + \log_2(x-2) \\ -\log_2(x-2) & -\log_2(x-2) \\ \hline 2\log_2 x - \log_2(x-2) = 3 & \end{array}$$

$$\log_2 x^2 - \log_2(x-2) = 3$$

$$\log_2\left(\frac{x^2}{x-2}\right) = 3$$

$$\frac{x^2}{x-2} = 2^3$$

$$\frac{x^2}{x-2} = 8$$

$$\frac{\cancel{x-2}}{1} \cdot \frac{x^2}{\cancel{x-2}} = 8(x-2)$$

$$x^2 = 8x - 16$$

$$\begin{array}{rr} x^2 & = 8x - 16 \\ -8x + 16 & -8x + 16 \\ \hline x^2 - 8x + 16 = 0 & \end{array}$$

$$(x-4)(x-4) = 0$$

$$\begin{array}{r} x - 4 = 0 \\ \underline{\quad +4 \quad +4} \\ x \;=\; 4 \end{array}$$

This solution works.