2.10 Quadratics in Disguise: Substitution

Sometimes, when an equation appears complicated, it can be rewritten in a more familiar way by substituting a simple variable for a more complicated expression. We often do this at the higher levels in mathematics to break a problem down into more "digestible" pieces. It helps us to focus on one thing at a time. For many of the following examples, you will see that you can either use substitution or a method that you have already learned, but you will also realize that substitution is often the easier method. If you decide to take a Calculus course at some point in the future, the ideas involved in this method will help you tremendously! The concept of breaking a problem down into smaller, more understandable pieces, is universal and it is a skill that can be applied to all kinds of problems outside of mathematics.

Examples

Solve each equation for the indicated variable.

1. $x^4 - 10x^2 + 9 = 0$

   Notice that the variable in the first term is the square of the variable in the second term. That is how we know we can write this as a quadratic.

   This equation could be written as:

$$(x^2)^2 - 10x^2 + 9 = 0$$

Let $y = x^2$

$$\Rightarrow \quad y^2 - 10y + 9 = 0$$

$$\Rightarrow \quad (y-9)(y-1)=0$$

$$\Rightarrow \quad y=9; \;\; y=1$$

Now, we need to find our original variable x by replacing y with x^2 in our solutions:

$$\Rightarrow \quad x^2=9; \;\; x^2=1$$

Solve each of these for x:

$$\Rightarrow \quad \sqrt{x^2}=\pm\sqrt{9}; \;\; \sqrt{x^2}=\pm\sqrt{1}$$

$$\Rightarrow \quad x=\pm 3; \;\; x=\pm 1$$

We have four solutions to this equation: $-3, -1, 1,$ and 3

Alternative method: We could also factor this one from the beginning to solve:

$x^4-10x^2+9=0$

$\Rightarrow (x^2-9)(x^2-1)=0$

$\Rightarrow (x+3)(x-3)(x+1)(x-1)=0$

$\Rightarrow x=-3, x=3, x=-1, x=1$

Not every equation we solve will have an alternative method, but when they do it is reassuring to know there is more than one way to get to the answer(s).

2. $3x-4x^{\frac{1}{2}}+1=0$

Notice again that the variable in the first term is the square of the variable in the second term. Another way to recognize this is that the power on the first term is double the power on the second term.

This equation could be written as:

$$3\left(x^{\frac{1}{2}}\right)^2-4x^{\frac{1}{2}}+1=0$$

Let $y=x^{\frac{1}{2}}$

$$\Rightarrow \quad 3y^2-4y+1=0$$

$$\Rightarrow \quad (3y-1)(y-1)=0$$

$$\Rightarrow \quad y=\frac{1}{3}; \;\; y=1$$

Now, we need to find our original variable x by replacing y with $x^{\frac{1}{2}}$ in our solutions:

$$\Rightarrow \quad x^{\frac{1}{2}}=\frac{1}{3}; \;\; x^{\frac{1}{2}}=1$$

Solve each of these for x:

$$\Rightarrow \quad \left(x^{\frac{1}{2}}\right)^2=\left(\frac{1}{3}\right)^2; \;\; \left(x^{\frac{1}{2}}\right)^2=1^2$$

$$\Rightarrow \quad x=\frac{1}{9}; \;\; x=1$$

Alternative method: We could also isolate the radical to solve this one:

$$3x-4x^{\frac{1}{2}}+1=0$$

$$\Rightarrow 3x-4\sqrt{x}+1=0$$

$$\Rightarrow 3x+1=4\sqrt{x}$$

$$\Rightarrow (3x+1)^2=\left(4\sqrt{x}\right)^2$$

$$\Rightarrow 9x^2+6x+1=16x$$

$$\Rightarrow 9x^2-10x+1=0$$

$$\Rightarrow (9x-1)(x-1)=0$$

$$\Rightarrow x=\frac{1}{9}, x=1$$

Substitution is quicker for this one…

We have to check our solutions because our original equation has a radical in it:

$x=\frac{1}{9}$	$x=1$
$3\cdot\frac{1}{9}-4\cdot\left(\frac{1}{9}\right)^{\frac{1}{2}}+1=0$	$3\cdot 1-4\cdot 1^{\frac{1}{2}}+1=0$
$\frac{3}{9}-4\sqrt{\frac{1}{9}}+1=0$	$3-4\sqrt{1}+1=0$
$\frac{1}{3}-4\cdot\frac{1}{3}+1=0$	$3-4\cdot 1+1=0$
$\frac{1}{3}-\frac{4}{3}+1=0$	$3-4+1=0$
$-\frac{3}{3}+1=0$ ✓	$-1+1=0$ ✓

We have two solutions to this equation: $\frac{1}{9}$ and 1

3. $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} + 12 = 0$

The best way to recognize that this is a quadratic is to check that the power on the first term is double the power on the second term.

This equation could be written as:

$$\left(x^{\frac{1}{3}}\right)^2 - 7x^{\frac{1}{3}} + 12 = 0$$

We do not have a good alternative method for this one, although you might recognize that you can factor this into $(x^{\frac{1}{3}} - 4)(x^{\frac{1}{3}} - 3) = 0$ and proceed. If so, that works, too. We believe substitution is easier.

$$\text{Let } y = x^{\frac{1}{3}}$$

$$\Rightarrow \quad y^2 - 7y + 12 = 0$$

$$\Rightarrow \quad (y-3)(y-4) = 0$$

$$\Rightarrow \quad y = 3; \;\; y = 4$$

Now, we need to find our original variable x by replacing y with $x^{\frac{1}{3}}$ in our solutions:

$$\Rightarrow \quad x^{\frac{1}{3}} = 3; \quad x^{\frac{1}{3}} = 4$$

Solve each of these for x by cubing both sides:

$$\Rightarrow \quad \left(x^{\frac{1}{3}}\right)^3 = 3^3; \; \left(x^{\frac{1}{3}}\right)^3 = 4^3$$

$$\Rightarrow \quad x = 27; \;\; x = 64$$

We should check our solutions because our original equation has a radical in it, although with a cube root everything should be fine:

$x = 27$ $\qquad\qquad\qquad\qquad$ $x = 64$

$27^{\frac{2}{3}} - 7 \cdot 27^{\frac{1}{3}} + 12 = 0$ $\qquad$ $64^{\frac{2}{3}} - 7 \cdot 64^{\frac{1}{3}} + 12 = 0$

$\left(\sqrt[3]{27}\right)^2 - 7\sqrt[3]{27} + 12 = 0$ $\qquad$ $\left(\sqrt[3]{64}\right)^2 - 7\sqrt[3]{64} + 12 = 0$

$9 - 7 \cdot 3 + 12 = 0$ $\qquad$ $16 - 7 \cdot 4 + 12 = 0$

$9 - 21 + 12 = 0$ $\qquad$ $16 - 28 + 12 = 0$

$-12 + 12 = 0$ ✓ $\qquad$ $-12 + 12 = 0$ ✓

We have two solutions to this equation: 27 and 64

4. $(r^2 + 2r)^2 - 2(r^2 + 2r) - 3 = 0$

Notice that the variable expression in the first term is the square of the variable expression in the second term. That is how we know we can write this as a quadratic.

$$(r^2 + 2r)^2 - 2(r^2 + 2r) - 3 = 0$$

Let $y = r^2 + 2r$

⇨ $y^2 - 2y - 3 = 0$

⇨ $(y - 3)(y + 1) = 0$

⇨ $y = 3;\ \ y = -1$

Now, we need to find our original variable r by replacing y with $r^2 + 2r$ in our solutions:

$$\Rightarrow \quad r^2 + 2r = 3; \;\; r^2 + 2r = -1$$

Solve each of these quadratics for r:

$$\Rightarrow \quad r^2 + 2r - 3 = 0; \;\; r^2 + 2r + 1 = 0$$

$$\Rightarrow \quad (r+3)(r-1) = 0; \;\; (r+1)^2 = 0$$

$$\Rightarrow \quad r = -3; \; r = 1; \;\; r = -1$$

We have three solutions to this equation: $-3, -1,$ and 1

Alternative method: We could also multiply this one out from the beginning to solve, but the grouping is a bit tricky when you factor:

$$(r^2 + 2r)^2 - 2(r^2 + 2r) - 3 = 0$$

$\Rightarrow r^4 + 4r^3 + 4r^2 - 2r^2 - 4r - 3 = 0$

$\Rightarrow r^4 + 4r^3 + 2r^2 - 4r - 3 = 0$

$\Rightarrow r^4 + 2r^2 - 3 + 4r^3 - 4r = 0$

$\Rightarrow (r^2 + 3)(r^2 - 1) + 4r(r^2 - 1) = 0$

$\Rightarrow (r^2 - 1)(r^2 + 4r + 3) = 0$

$\Rightarrow (r+1)(r-1)(r+3)(r+1) = 0$

$\Rightarrow r = -1, \; r = 1, r = -3$

Substitution is definitely a faster way to solve this one. (There is also another method involving the Rational Root Theorem that you will learn in a future course.)

5. $2(2x+1)^{\frac{2}{3}} - 7(2x+1)^{\frac{1}{3}} = -6$

The best way to recognize that this is a quadratic is to check that the power on the first term is double the power on the second term. $\left(\frac{2}{3} \text{ is double } \frac{1}{3}\right)$

This equation could be written as:

$$2\left[(2x+1)^{\frac{1}{3}}\right]^2 - 7(2x+1)^{\frac{1}{3}} = -6$$

$$\text{Let } y = (2x+1)^{\frac{1}{3}}$$

$$\Rightarrow \quad 2y^2 - 7y = -6$$

We do not have a good alternative method for this one, although we do want to remind our students that exponents do NOT distribute over addition, so you cannot distribute the $\frac{2}{3}$ $\left(\text{or the } \frac{1}{3}\right)$ to the terms inside the parenthesis.

$$\Rightarrow \quad 2y^2 - 7y + 6 = 0$$

$$\Rightarrow \quad (2y-3)(y-2) = 0$$

$$\Rightarrow \quad y = \frac{3}{2}; \;\; y = 2$$

Now, we need to find our original variable x by replacing y with $(2x+1)^{\frac{1}{3}}$ in our solutions:

$$\Rightarrow \quad (2x+1)^{\frac{1}{3}} = \frac{3}{2}; \quad (2x+1)^{\frac{1}{3}} = 2$$

Solve each of these for x by cubing both sides:

$2x + 1 = \frac{27}{8}$

$\Rightarrow 8(2x+1) = 27$

$\Rightarrow 16x + 8 = 27$

$\Rightarrow 16x = 19$

$\Rightarrow x = \frac{19}{16}$

$$\Rightarrow \quad \left((2x+1)^{\frac{1}{3}}\right)^3 = \left(\frac{3}{2}\right)^3; \;\; \left((2x+1)^{\frac{1}{3}}\right)^3 = 2^3$$

$$\Rightarrow \quad 2x + 1 = \frac{27}{8}; \quad 2\,x + 1 = 8$$

$$\Rightarrow \quad x = \frac{19}{16}; \quad x = \frac{7}{2}$$

We should check our solutions because our original equation has rational exponents in it, although with a cube root everything should be fine. It is simplest if we compute the inside expression $2x+1$ for each value of x, and then insert our result into the original equation to replace the expression. This is easy because we already have it in our work above. When $x = \frac{19}{16}, \; 2x + 1 = \frac{27}{8}$ and when $x = \frac{7}{2}, 2\,x + 1 = 8$:

$x = \frac{19}{16}$ $\qquad\qquad$ $x = \frac{7}{2}$

$$2\left(\frac{27}{8}\right)^{\frac{2}{3}} - 7 \cdot \left(\frac{27}{8}\right)^{\frac{1}{3}} = -6$$

$$2\left(\sqrt[3]{\frac{27}{8}}\right)^2 - 7\sqrt[3]{\frac{27}{8}} = -6$$

$$2\left(\frac{3}{2}\right)^2 - 7 \cdot \frac{3}{2} = -6$$

$$\frac{18}{4} - \frac{21}{2} = -6$$

$$\frac{9}{2} - \frac{21}{2} = -\frac{12}{2} = -6$$

$$2 \cdot 8^{\frac{2}{3}} - 7 \cdot 8^{\frac{1}{3}} = -6$$

$$2(\sqrt[3]{8})^2 - 7\sqrt[3]{8} = -6$$

$$2 \cdot 2^2 - 7 \cdot 2 = -6$$

$$2 \cdot 4 - 14 = -6$$

$$8 - 14 = -6$$

We have two solutions to this equation: $\frac{19}{16}$ and $\frac{7}{2}$

6. $15a^{-2} - 8a^{-1} + 1 = 0$

This equation could be written as:

$$15(\,a^{-1})^2 - 8a^{-1} + 1 = 0$$

Let $y = a^{-1}$

⇨ $15y^2 - 8y + 1 = 0$

⇨ $(3y-1)(5y-1) = 0$

⇨ $y = \frac{1}{3};\ \ y = \frac{1}{5}$

Alternative method: We could also write this one with denominators and solve it as a rational equation (find LCD):

$15a^{-2} - 8a^{-1} + 1 = 0$

⇨ $\frac{15}{a^2} - \frac{8}{a} + 1 = 0$

⇨ $\frac{15}{a^2} - \frac{8a}{a^2} + \frac{a\text{^}2}{a^2} = \frac{0}{a^2}$

⇨ $15 - 8a + a^2 = 0$

⇨ $(3-a)(5-a) = 0$

⇨ $a = 3, a = 5$

Now, we need to find our original variable a by replacing y with a^{-1} in our solutions:

$$\Rightarrow \quad a^{-1} = \frac{1}{3}; \;\; a^{-1} = \frac{1}{5}$$

Solve each of these for a:

$$\Rightarrow \quad (a^{-1})^{-1} = \left(\frac{1}{3}\right)^{-1}; \;\; (a^{-1})^{-1} = \left(\frac{1}{5}\right)^{-1}$$

$$\Rightarrow \quad a = 3; \;\; a = 5$$

We have two solutions to this equation: 3 and 5

7. $9\left(\frac{2t+7}{t}\right)^2 - 42\left(\frac{2t+7}{t}\right) + 49 = 0$

Let $y = \frac{2t+7}{t}$

$$\Rightarrow \quad 9y^2 - 42y + 49 = 0$$

$$\Rightarrow \quad (3y - 7)(3y - 7) = 0$$

$$\Rightarrow \quad y = \frac{7}{3}$$

Now, we need to find our original variable t by replacing y with $\frac{2t+7}{t}$ in our solution:

$$\Rightarrow \quad \frac{2t+7}{t} = \frac{7}{3}$$

Solve for t by cross multiplying:

➩ $3(2t+7) = 7t$

➩ $6t + 21 = 7t$

➩ $21 = t$

➩ $t = 21$

We have one solution to this equation: 21

8. $2(y+1)^2 - 5(y+1) = -1$

Since the variable y is already in use here, we will use a different variable for our substitution. (It is important not to use the same variable for the substitution as the variable in the equation as this will lead to false statements and confusion!)

Let $u = y + 1$

➩ $2u^2 - 5u = -1$

➩ $2u^2 - 5u + 1 = 0$

This does not factor, so we need to use the quadratic formula to solve for u:

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 2, b = -5, and\ c = 1$

$$u = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{5 \pm \sqrt{17}}{4}$$

Now, we need to find our original variable y by replacing u with $y + 1$ in our solution:

⇨ $y + 1 = \frac{5 \pm \sqrt{17}}{4}$

Solve for y by subtracting 1 from both sides and combing with a common denominator:

⇨ $y = \frac{5 \pm \sqrt{17}}{4} - 1$

⇨ $y = \frac{5 \pm \sqrt{17}}{4} - \frac{4}{4}$

⇨ $y = \frac{5 \pm \sqrt{17} - 4}{4}$

⇨ $y = \frac{1 \pm \sqrt{17}}{4}$

We have two solutions to this equation:

$\frac{1+\sqrt{17}}{4} \approx 1.28$ and $\frac{1-\sqrt{17}}{4} \approx -0.78$

Can you do this problem an alternate way? Which way seems simpler to you?

9. $4x^6 + 28x^3 = 32$

Let $u = x^3$

⇨ $4u^2 + 28u = 32$

⇨ $4u^2 + 28u - 32 = 0$

Factoring the polynomial, we obtain:

⇨ $4(u^2 + 7u - 8) = 0$

⇨ $4(u + 8)(u - 1) = 0$

⇨ $u + 8 = 0; u - 1 = 0$

⇨ $u = -8;\ \ u = 1$

Back substituting, we have:

⇨ $x^3 = -8;\ \ x^3 = 1$

If we were to just take the cube root of both sides of these equations, we would get the real solutions only, but not the complex solutions. Instead, we bring everything to one side and factor:

⇨ $x^3 + 8 = 0;\ \ x^3 - 1 = 0$

We need to factor these using the sum and difference of cubes formulas to obtain all solutions:

$$\Rightarrow \quad (x+2)(x^2-2x+4)=0; \;\; (x-1)(x^2+x+1)=0$$

If we use the quadratic formula on the second piece of each equation, we will arrive at the six solutions:

$$\Rightarrow \quad x=-2; \;\; x=1\pm\sqrt{3}; x=1; x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$