2.9 The Quadratic Formula

The standard form for a quadratic equation is as follows:

$$ax^2 + bx + c = 0\,; \qquad a > 0$$

What would happen if we completed the square on a generic quadratic in standard form? We should obtain a generic solution that will work for all quadratics. That is exactly what happens and we call this solution the Quadratic Formula:

$$ax^2 + bx + c = 0$$

First, factor out a and divide both sides by it so that we can complete the square.

$$\Rightarrow \quad a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$$

$$\Rightarrow \quad \frac{a(x^2+\frac{b}{a}x+\frac{c}{a})}{a} = \frac{0}{a}$$

Now, we need to take the coefficient of x, which is $\frac{b}{a}$, and double the denominator and square it. Then add and subtract it to complete the square.

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^2 - \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^2 + \frac{c}{a} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$-\frac{b^2}{4a^2} + \frac{c}{a}$$
$$= \frac{-b^2}{4a^2} + \frac{c \cdot 4a}{a \cdot 4a}$$
$$= \frac{-b^2}{4a^2} + \frac{4ac}{4a^2}$$
$$= \frac{-b^2 + 4ac}{4a^2}$$
$$= \frac{-(b^2 - 4ac)}{4a^2}$$
$$= -\frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad \sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\Rightarrow \qquad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

And *viola*!! We now have a formula that will solve any quadratic in standard form!!

To use the quadratic formula, we must sometimes put our equation in standard form first so that we can read off the coefficients and plug them into the formula. Notice that the leading term must be positive. That is important to check when you are using the formula to solve problems.

It is also important to note that, even though we can always use the quadratic formula, it is usually easier to factor when possible. Take the example in the last section that we could have factored and use the quadratic formula to solve it instead: $2x^2 - 5x + 2 = 0$

Here $a = 2, \ b = -5, \ c = 2$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-(-5) \pm \sqrt{(-5)^2-4 \cdot 2 \cdot 2}}{2 \cdot 2}$$

$$\Rightarrow \qquad x = \frac{5 \pm 3}{4}$$

$$\Rightarrow \qquad x = \frac{5+3}{4} = \frac{8}{4} = 2 \quad \text{or} \quad x = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2}$$

Work on the radical first:

$= \sqrt{(-5)^2 - 4 \cdot 2 \cdot 2} = \sqrt{25 - 16} = \sqrt{9}$

But $\sqrt{9} = 3$

You can see that using the quadratic formula is just a shortcut for completing the square, but factoring would still yield those answers in fewer steps. A timesaving technique to solve a quadratic is to try to factor first and if you can't then use the quadratic formula.

<u>Examples</u>

Solve each equation for the indicated variable.

1. $2x^2 + 4x + 1 = 0$

Here $a = 2,\ b = 4,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{8}}{4}$$

$$\Rightarrow \quad x = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2}$$

$$\left(or -1 \pm \frac{\sqrt{2}}{2} \right)$$

Be careful not to cancel over addition. These 4's do not cancel, but you can factor out 2 from the numerator and cancel that with the denominator.

$$\frac{-4 \pm 2\sqrt{2}}{4} = \frac{2\left(-2 \pm \sqrt{2}\right)}{4} = \frac{-2 \pm \sqrt{2}}{2}$$

2. $x^2 + 6x + 4 = 0$

Here $a = 1,\ b = 6,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

Simplify your radical first:

$\sqrt{6^2 - 4 \cdot 1 \cdot 4} = \sqrt{36 - 16} = \sqrt{20}$

And $\sqrt{20} = \sqrt{2 \cdot 2 \cdot 5} = 2\sqrt{5}$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{20}}{2}$$

$$\Rightarrow \quad x = \frac{-6 \pm 2\sqrt{5}}{2} = -3 \pm \sqrt{5}$$

Note:

$\frac{-6 \pm 2\sqrt{5}}{2} = \frac{2(-3 \pm \sqrt{5})}{2} = -3 \pm \sqrt{5}$

3. $y^2 + 10y - 7 = 0$

Here $a = 1,\ b = 10,\ c = -7$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad y = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot (-7)}}{2 \cdot 1}$$

$$\Rightarrow \quad y = \frac{-10 \pm \sqrt{128}}{2}$$

Pay attention to signs when simplifying your radical:

$\sqrt{10^2 - 4 \cdot 1 \cdot (-7)} = \sqrt{100 + 28} = \sqrt{128}$

And $\sqrt{128} = \sqrt{2 \cdot 64} = 8\sqrt{2}$

$$\Rightarrow \quad y = \frac{-10 \pm 8\sqrt{2}}{2} = -5 \pm 4\sqrt{2}$$

4. $x^2 + 6x = -1$

First, we need to get 0 on one side. If we don't do this, we won't be able to see the correct signs on our coefficients a, b and c.

$$x^2 + 6x + 1 = 0$$

Here $a = 1,\ b = 6,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{32}}{2}$$

$$\Rightarrow \quad x = \frac{-6 \pm 4\sqrt{2}}{2} = -3 \pm 2\sqrt{2}$$

5. $\frac{x^2}{2} = -\frac{5}{2}x - 1$

Bringing everything to one side, we get

$$\frac{x^2}{2} + \frac{5}{2}x + 1 = 0 \ \ or \ \ \frac{1}{2}x^2 + \frac{5}{2}x + 1 = 0$$

While we could go ahead and use the quadratic formula with $a = \frac{1}{2},\ b = \frac{5}{2},\ c = 1$, we could also save time by clearing the fractions - multiplying both sides by the common denominator 2 (since we have an equation, as long as we multiply both sides by the same number, we haven't changed anything but form).

$$2\left(\frac{x^2}{2} + \frac{5}{2}x + 1\right) = 0 \cdot 2$$

$$\Rightarrow \qquad x^2 + 5x + 2 = 0$$

Much simpler, right?

Here $a = 1,\ b = 5,\ c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \qquad x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$\Rightarrow \qquad x = \frac{-5 \pm \sqrt{17}}{2}$$

6. $x(x + 5) = 5$

We need to put this in standard form first, so distribute x and then subtract 5 from both sides:

$$x^2 + 5x = 5$$

$$\Rightarrow \quad x^2 + 5x - 5 = 0$$

Some students are tempted to set each factor equal to 5 to solve this equation, but that technique only works when you have 0 on the other side. Two numbers can multiply to be 5 without either of them being 5 (e.g. $\frac{1}{2} \cdot 10 = 5$, but neither factor is 5.) This not the case with 0, though. If two numbers multiply to be 0, then one of them has to be 0.

Here $a = 1,\ b = 5,\ c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1}$$

Reminder: Be careful with the signs inside of the radical.

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{45}}{2}$$

$$\Rightarrow \quad x = \frac{-5 \pm 3\sqrt{5}}{2}$$

7. $-9h^2 + 6h = 7$

$$-9h^2 + 6h - 7 = 0$$

Notice that the leading term is negative. We must multiply (or divide) both sides by -1 to get a positive leading term if we want to use the quadratic formula:

$$-1 \cdot (-9h^2 + 6h - 7) = 0 \cdot (-1)$$

$$\Rightarrow \quad 9h^2 - 6h + 7 = 0$$

Here $a = 9,\ b = -6,\ c = 7$

$$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad h = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 9 \cdot 7}}{2 \cdot 9}$$

Pay attention to signs when simplifying your radical:

$$\sqrt{(-6)^2 - 4 \cdot 9 \cdot 7} = \sqrt{36 - 252} = \sqrt{-216}$$

And $\sqrt{-216} = i\sqrt{216} = 6i\sqrt{6}$

Do a factor tree on 216 to see this….

$$\Rightarrow \quad h = \frac{6 \pm \sqrt{-216}}{18}$$

$$\Rightarrow \quad h = \frac{6 \pm 6i\sqrt{6}}{18} = \frac{1 \pm i\sqrt{6}}{3}$$

Reminder: Be careful not to cancel over addition. These 6 and the 18 do not cancel, but you can factor out 6 from the numerator and cancel that with the denominator.

$$\frac{6 \pm 6i\sqrt{6}}{18} = \frac{6\left(1 \pm i\sqrt{6}\right)}{18} = \frac{1 \pm i\sqrt{6}}{3}$$

Since this is a complex number, we should write it in the form $a + bi$.

$$\Rightarrow \quad h = \frac{1}{3} \pm \frac{\sqrt{6}}{3} i$$