2.8 The Square Root Property and Completing the Square

The Square Root Property is something we have already been using to solve equations involving squares, but now we will formalize it. Each time we "undo" a square by taking the square root of both sides of an equation, we are using this property. For example, to solve the equation $x^2 = 4$, we could take the square root of both sides $\sqrt{x^2} = \pm\sqrt{4}$ and this would give us both answers $x = \pm 2$ (meaning $x = 2, x = -2$).

Square Root Property: *For any non-negative real number 'c', if* $x^2 = c$, *then* $x = \sqrt{c}$ *or* $x = -\sqrt{c}$.

<u>Examples</u>

Solve each equation for the indicated variable.

1. $x^2 = 9$

Using the square root property to solve this equation, we will take the square root of both sides.

$$x^2 = 9$$

$$\Rightarrow \quad \sqrt{x^2} = \pm\sqrt{9}$$

$$\Rightarrow \quad x = \pm 3$$

> Recall that we can also solve this problem by factoring:
>
> $x^2 = 9 \Rightarrow x^2 - 9 = 0 \Rightarrow (x+3)(x-3) = 0$
>
> $\Rightarrow x + 3 = 0;\ x - 3 = 0 \Rightarrow x = -3,\ x = 3$
>
> Clearly, using the square root property has its advantages. It will also lead us to a method to solve those quadratics that will not factor (using the quadratic formula) in the next lesson.

(This means we have two answers: $x = 3, x = -3$)

2. $x^2 - 12 = 0$

$$\begin{array}{r} x^2 - 12 = 0 \\ +12 \quad +12 \\ \hline \end{array}$$

⇨ $x^2 = 12$

⇨ $\sqrt{x^2} = \pm\sqrt{12} = \pm\sqrt{2 \cdot 2 \cdot 3}$

⇨ $x = \pm 2\sqrt{3}$

Alternative method:

$x^2 = 12$ ⇨ $x^2 - 12 = 0$

⇨ $\left(x + \sqrt{12}\right)\left(x - \sqrt{12}\right) = 0$

⇨ $x + \sqrt{12} = 0;\ x - \sqrt{12} = 0$

⇨ $x = -\sqrt{12} = -2\sqrt{3},$

$x = \sqrt{12} = 2\sqrt{3}$

3. $x^2 + 25 = 0$

$$\begin{array}{r} x^2 + 25 = 0 \\ -25 \quad -25 \\ \hline \end{array}$$

⇨ $x^2 = -25$

⇨ $\sqrt{x^2} = \pm\sqrt{-25} = \pm i\sqrt{25}$

⇨ $x = \pm 5i$

Alternative method:

$x^2 + 25 = 0$ ⇨ $x^2 - (-25) = 0$

⇨ $\left(x + \sqrt{-25}\right)\left(x - \sqrt{-25}\right) = 0$

⇨ $(x + 5i)(x - 5i) = 0$

⇨ $x + 5i = 0;\ x - 5i = 0$

⇨ $x = -5i,\ x = 5i$

4. $9x^2 + 4 = 0$

$$\begin{array}{r} 9x^2 + 4 = 0 \\ -4 \quad -4 \\ \hline \end{array}$$

⇨ $9x^2 \quad = -4$

Alternative method:

$9x^2 + 4 = 0$ ⇨ $9x^2 - (-4) = 0$

⇨ $\left(3x + \sqrt{-4}\right)\left(3x - \sqrt{-4}\right) = 0$

⇨ $(3x + 2i)(3x - 2i) = 0$

⇨ $3x + 2i = 0;\ 3x - 2i = 0$

⇨ $x = -\frac{2}{3}i,\ x = \frac{2}{3}i$

$$\Rightarrow \quad 9x^2 = -4$$

$$\Rightarrow \quad x^2 = -\frac{4}{9}$$

$$\Rightarrow \quad \sqrt{x^2} = \pm\sqrt{-\frac{4}{9}} = \pm i\sqrt{\frac{4}{9}}$$

$$\Rightarrow \quad x = \pm\frac{2}{3}i$$

> You can also take the square root at this step since radicals distribute over multiplication rather than isolating the radical completely. You would get $\sqrt{9x^2} = \pm\sqrt{-4} \Rightarrow \quad 3x = \pm 2i$ and then divide both sides by 3.

Notice that in every example, we have isolated the square of a variable in order to "undo" it. We can allow for more complicated squares involving the variable and the process will remain the same. For example, instead of just x^2, we could have something like $(x-5)^2$ or $(4x+1)^2$. (When this is the case, there are more options for solving since you could also square it and bring everything to one side and factor or you could also use the difference of squares method as before.)

5. $(x-1)^2 = 16$

$$(x-1)^2 = 16$$

$$\Rightarrow \quad \sqrt{(x-1)^2} = \pm\sqrt{16}$$

$$\Rightarrow \quad x - 1 = \pm 4$$

$$\Rightarrow \quad x - 1 = 4 \text{ or } x - 1 = -4$$

$$\Rightarrow \quad x = 5 \text{ or } x = -3$$

> Alternative methods:
>
> 1. $(x-1)^2 = 16$
> $\Rightarrow (x-1)^2 - 16 = 0$
> $\Rightarrow ((x-1)+4)((x-1)-4) = 0$
> $\Rightarrow (x+3)(x-5) = 0$
> $\Rightarrow x = -3,\ x = 5$
>
> 2. $(x-1)^2 = 16$
> $\Rightarrow x^2 - 2x + 1 = 16$
> $\Rightarrow x^2 - 2x - 15 = 0$
> $\Rightarrow (x-5)(x+3) = 0$
> $\Rightarrow x = -3,\ x = 5$

From this point forward in this section, we will omit the alternative methods and simply focus on using the square root property to solve the equations. The point of showing the alternative methods was just to connect the new ideas to the concepts you know and to reassure our students that all valid methods lead to the same answer.

6. $(x+2)^2 - 8 = 36$

$$(x+2)^2 - 8 = 36$$
$$\underline{\qquad\quad +8 \quad +8}$$
$$\Rightarrow \quad (x+2)^2 = 44$$

We must first isolate the square by adding 8 to both sides.

$$\Rightarrow \quad \sqrt{(x+2)^2} = \pm\sqrt{44}$$

$$\Rightarrow \quad x + 2 = \pm 2\sqrt{11}$$
$$\underline{-2 \qquad\qquad -2}$$
$$\Rightarrow \quad x = -2 \pm 2\sqrt{11}$$

$$\Rightarrow \quad x = -2 + 2\sqrt{11} \ \ or \ \ x = -2 - 2\sqrt{11}$$

7. $(3x-1)^2 - 5 = 31$

$$(3x-1)^2 - 5 = 31$$
$$\underline{\qquad\quad +5 \quad +5}$$
$$\Rightarrow \quad (3x-1)^2 = 36$$

We must first isolate the square by adding 5 to both sides.

$$\Rightarrow \quad \sqrt{(3x-1)^2} = \pm\sqrt{36}$$

$$\Rightarrow \quad 3x - 1 = \pm 6$$
$$\Rightarrow \quad 3x - 1 = 6 \ \ or \ \ 3x - 1 = -6$$
$$\Rightarrow \quad x = \frac{7}{3} \ \ or \ \ x = -\frac{5}{3}$$

But what if you don't have a perfect square on one side of the equation? For example, consider the equation $x^2 + 10x + 8 = 0$. We can't factor this, so what do we do? It turns out we could rewrite the left side of the equation as a perfect square (using a method called *completing the square*) and then we can proceed using the square root property as we did in prior examples.

Completing the Square

In order to complete the square on an *expression*, like $x^2 + 10x + 8$, we would need to add something to it to get a perfect square but also subtract the same thing to it so that we do not change it. When we are completing the square on an equation, like $x^2 + 10x + 8 = 0$, we can either work on one side as an expression or we can add the same thing to both sides. Let's begin by completing the square on the expression $x^2 + 10x + 8$:

Start by thinking about what you can add to the first two terms to make something that will factor into a perfect square. If you think about it, you should come up with 25, since $x^2 + 10x + 25 = (x+5)^2$, a perfect square. You can get this quickly by taking half of the coefficient of x and squaring it. But if we add it, we must also subtract it:

$$x^2 + 10x + 8 = x^2 + 10x \mathbf{+ 25 - 25} + 8$$

Now group the first three terms and factor into a perfect square and group the last two terms and add them together. Suddenly, you have a perfect square that can be isolated if it is on one side of an equation!

$$= (x+5)^2 - 17$$

Note: The general rule for completing the square on the expression $x^2 + bx + c$ is to add and subtract $\left(\frac{b}{2}\right)^2$. (This only works when the coefficient of x^2 is 1. If there is any other number in front of x^2, you must first factor it out before completing the square. We will see examples of this in a while.)

8. $x^2 + 10x + 8 = 0$

$$\Rightarrow \quad x^2 + 10x \mathbf{+ 25 - 25} + 8 = 0$$

$$\Rightarrow \quad (x+5)^2 - 17 = 0$$

$$\Rightarrow \quad (x+5)^2 = 17$$

$$\Rightarrow \quad \sqrt{(x+5)^2} = \pm\sqrt{17}$$

$$\Rightarrow \quad x + 5 = \pm\sqrt{17}$$

$$\underline{\quad -5 \qquad\quad -5 \quad}$$

$$\Rightarrow \quad x = -5 \pm \sqrt{17}$$

We complete the square on one side of the equation, as we did above, adding and subtracting 25:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = 5^2 = 25$$

Then group the first three terms as a perfect square and add together the last two terms.

Isolate the square, as usual, and take the square root of both sides.

We could have done the same problem by adding 25 to both sides instead of adding and subtracting 25 to the same side as follows:

$$x^2 + 10x + 8 = 0$$

$$\Rightarrow \quad x^2 + 10x = -8$$

$$\Rightarrow \quad x^2 + 10x \mathbf{+ 25} = -8 \mathbf{+ 25}$$

$$\Rightarrow \quad (x+5)^2 = 17$$

This method is sometimes more convenient, but often you will only have one side to work with when you are graphing later, so we choose to present it by adding and subtracting to one side rather than adding to both sides in order for you to get experience doing it this way prior to graphing parabolas.

$$\Rightarrow \quad \sqrt{(x+5)^2} = \pm\sqrt{17}$$

$$\Rightarrow \quad x+5 = \pm\sqrt{17}$$

$$\underline{\quad -5 \qquad\qquad -5 \quad}$$

$$\Rightarrow \quad x = -5 \pm \sqrt{17}$$

9. $x^2 + 8x + 6 = 0$

$$\Rightarrow \quad x^2 + 8x \mathbf{+ 16 - 16} + 6 = 0$$

Note: $\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 4^2 = 16$

$$\Rightarrow \quad (x+4)^2 - 10 = 0$$

$$\Rightarrow \quad (x+4)^2 = 10$$

$$\Rightarrow \quad \sqrt{(x+4)^2} = \pm\sqrt{10}$$

$$\Rightarrow \quad x+4 = \pm\sqrt{10}$$

$$\underline{\quad -4 \qquad\qquad -4 \quad}$$

$$\Rightarrow \quad x = -4 \pm \sqrt{10}$$

10. $3x^2 + 24x + 3 = 0$

Notice that the coefficient of x^2 is not 1 in this example. Our method of completing the square on $ax^2 + bx + c$ will only work when we have $a = 1$ because $\left(\frac{b}{2}\right)^2$ will only complete the square for $x^2 + bx$. The reason for this is that when we square $x + \frac{b}{2}$, we will always get $\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right) = x^2 + bx + \left(\frac{b}{2}\right)^2$. Therefore, we must factor out the coefficient of x^2 before we begin completing the square.

$$3(x^2 + 8x + 1) = 0$$

$\Rightarrow \quad \dfrac{3(x^2+8x+1)}{3} = \dfrac{0}{3}$

$\Rightarrow \quad x^2 + 8x + 1 = 0$

$\Rightarrow \quad x^2 + 8x \mathbf{+ 16 - 16} + 1 = 0$

$\Rightarrow \quad (x + 4)^2 - 15 = 0$

$\Rightarrow \quad (x + 4)^2 = 15$

$\Rightarrow \quad \sqrt{(x + 4)^2} = \pm\sqrt{15}$

$\Rightarrow \quad x + 4 = \pm\sqrt{15}$

$\quad\quad - 4 \quad\quad - 4$

$\Rightarrow \quad x = -4 \pm \sqrt{15}$

Since this is an equation, we can divide both sides by the coefficient we pulled out. If this were just an expression, however, we would need to keep the 3 outside until we completed the process and then distribute it to both pieces inside:

$3(x^2 + 8x + 1)$

$\Rightarrow 3(x^2 + 8x + \mathbf{16} - \mathbf{16} + 1)$

$\Rightarrow 3[(x + 4)^2 - 15]$

$\Rightarrow 3(x + 4)^2 - 45$

This will be an important distinction when we are using this process to put quadratics into graphing form a few lessons ahead…

If we had chosen to work this problem by adding to both sides, we would have to be careful to multiply by the number we pulled out before adding to the other side:

$3x^2 + 24x = -3$

$\Rightarrow 3(x^2 + 8x) = \mathbf{-3}$

$\Rightarrow 3(x^2 + 8x + \mathbf{16}) = -3 + \mathbf{48}$

$\Rightarrow 3(x + 4)^2 = 45$

$\Rightarrow (x + 4)^2 = 15$ etc.…

11. $\quad 2x^2 - 5x = -2$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad 2(x^2 - \frac{5}{2}x + 1) = 0$

First, we added 2 to both sides to get 0 on the other side. We do this so that we can work on one side of the equation and when we divide both sides by 2, we will still have 0 on the other side. Remember that it is not the only way to approach the problem.

Note that fractions can appear when factoring out the coefficient of x^2.

When adding or subtracting fractions, don't forget that you need a common denominator:

$$-\frac{25}{16}+1$$

$$=-\frac{25}{16}+\frac{1}{1}$$

$$=-\frac{25}{16}+\frac{16}{16}$$

$$=\frac{-25+16}{16}$$

$$=-\frac{9}{16}$$

$$\Rightarrow \quad \frac{2\left(x^2-\frac{5}{2}x+1\right)}{2}=\frac{0}{2}$$

$$\Rightarrow \quad x^2-\frac{5}{2}x+1=0$$

$$\Rightarrow \quad x^2-\frac{5}{2}x+\mathbf{\frac{25}{16}}-\mathbf{\frac{25}{16}}+1=0$$

$$\Rightarrow \quad \left(x-\frac{5}{4}\right)^2-\frac{9}{16}=0$$

$$\Rightarrow \quad \left(x-\frac{5}{4}\right)^2=\frac{9}{16}$$

$$\Rightarrow \quad \sqrt{\left(x-\frac{5}{4}\right)^2}=\pm\sqrt{\frac{9}{16}}$$

$$\Rightarrow \quad x-\frac{5}{4}=\pm\frac{3}{4}$$

$$\qquad +\frac{5}{4} \qquad\qquad +\frac{5}{4}$$

$\Rightarrow \quad x=\frac{5}{4}\pm\frac{3}{4}$ and we can simplify each of these:

$$\Rightarrow \quad x=\frac{5}{4}+\frac{3}{4}=\frac{8}{4}=2 \text{ or } \quad x=\frac{5}{4}-\frac{3}{4}=\frac{2}{4}=\frac{1}{2}$$

So the answers are $x=\frac{1}{2},\ x=2$

To get $\frac{b}{2}$, remember that dividing by 2 is the same as multiplying by $\frac{1}{2}$:

$$\left(\frac{b}{2}=\frac{\frac{5}{2}}{2}=\frac{\frac{5}{2}}{\frac{2}{1}}=\frac{5}{2}\cdot\frac{1}{2}=\frac{5}{4}\right)$$

The shortcut is to simply multiply the denominator by 2.

So if you have a fraction for b, just double the denominator and square your result to get $\left(\frac{b}{2}\right)^2$.

Here we get: $\left(\frac{b}{2}\right)^2=\left(\frac{5}{4}\right)^2=\frac{25}{16}$

When our answers are this nice (rational numbers – no radicals in them), we should be suspicious that there was an easier way to solve the equation. This is true because we could have factored to get the same answers quite quickly…

$$2x^2 - 5x = -2$$

$$\Rightarrow \quad 2x^2 - 5x + 2 = 0$$

$$\Rightarrow \quad (2x - 1)(x - 2) = 0$$

$$\Rightarrow \quad 2x - 1 = 0;\ x - 2 = 0$$

$$\Rightarrow \quad x = \frac{1}{2};\ x = 2$$

The lesson here is that you should always try to factor a quadratic before proceeding with a more complicated method. If you cannot factor, then completing the square (or using the quadratic formula that we learn about in the next section) is the way to go.

Let's look at a couple more examples that have fractions involved just to make sure that the fractions are not an issue. At this level, we need to be able to deal with fractions without any trouble.

12. $4x^2 - 5x = -3$

$\Rightarrow \quad 4x^2 - 5x + 3 = 0$

First, we added 3 to both sides to get 0 on the other side.

$\Rightarrow \quad 4(x^2 - \frac{5}{4}x + \frac{3}{4}) = 0$

$\Rightarrow \quad \frac{4(x^2 - \frac{5}{4}x + \frac{3}{4})}{4} = \frac{0}{4}$

$\Rightarrow \quad x^2 - \frac{5}{4}x + \frac{3}{4} = 0$

To get $\left(\frac{b}{2}\right)^2$, just double the denominator and square your result:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{5}{8}\right)^2 = \frac{25}{64}$$

$\Rightarrow \quad x^2 - \frac{5}{4}x + \mathbf{\frac{25}{64}} - \mathbf{\frac{25}{64}} + \frac{3}{4} = 0$

When adding or subtracting fractions, don't forget that you need a common denominator:

$$-\frac{25}{64} + \frac{3}{4}$$

$$= -\frac{25}{64} + \frac{3 \cdot 16}{4 \cdot 16}$$

$$= -\frac{25}{64} + \frac{3 \cdot 16}{4 \cdot 16}$$

$$= -\frac{25}{64} + \frac{48}{64}$$

$$= \frac{-25 + 48}{64}$$

$$= \frac{23}{16}$$

$\Rightarrow \quad \left(x - \frac{5}{8}\right)^2 + \frac{23}{64} = 0$

$\Rightarrow \quad \left(x - \frac{5}{8}\right)^2 = -\frac{23}{64}$

$\Rightarrow \quad \sqrt{\left(x - \frac{5}{8}\right)^2} = \pm\sqrt{-\frac{23}{64}}$

$\Rightarrow \quad x - \frac{5}{8} = \pm\frac{\sqrt{23}}{8}i$

$\quad +\frac{5}{8} \qquad +\frac{5}{8}$

$\Rightarrow \quad x = \frac{5}{8} \pm \frac{\sqrt{23}}{8}i$

13. $3x^2 = 10x + 9$

$$\Rightarrow \quad 3x^2 - 10x - 9 = 0$$

First, we brought everything to one side, subtracting 10x and 9 to both sides.

$$\Rightarrow \quad 3\left(x^2 - \frac{10}{3}x - 3\right) = 0$$

$$\Rightarrow \quad \frac{3\left(x^2 - \frac{10}{3}x - 3\right)}{3} = \frac{0}{3}$$

$$\Rightarrow \quad x^2 - \frac{10}{3}x - 3 = 0$$

To get $\left(\frac{b}{2}\right)^2$, just double the denominator and square your result:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{6}\right)^2 = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$$

Note that we reduced our fraction before we squared it. It makes sense to reduce fractions whenever you can since this results in smaller numbers that are easier to work with.

$$\Rightarrow \quad x^2 - \frac{10}{3}x + \mathbf{\frac{25}{9}} - \mathbf{\frac{25}{9}} - 3 = 0$$

When adding or subtracting fractions, don't forget that you need a common denominator:

$$-\frac{25}{9} - 3$$

$$= -\frac{25}{9} - \frac{3}{1}$$

$$= -\frac{25}{9} - \frac{3 \cdot 9}{1 \cdot 9}$$

$$= -\frac{25}{9} - \frac{27}{9}$$

$$= \frac{-25 - 27}{9}$$

$$= -\frac{52}{9}$$

$$\Rightarrow \quad \left(x - \frac{5}{3}\right)^2 - \frac{52}{9} = 0$$

$$\Rightarrow \quad \left(x - \frac{5}{3}\right)^2 = \frac{52}{9}$$

$$\Rightarrow \quad \sqrt{\left(x - \frac{5}{3}\right)^2} = \pm\sqrt{\frac{52}{9}}$$

$$\Rightarrow \quad x - \frac{5}{3} = \pm\frac{\sqrt{52}}{3} = \pm\frac{2\sqrt{13}}{3}$$

$$+\frac{5}{3} \qquad\qquad\qquad\qquad +\frac{5}{3}$$

$$\Rightarrow \quad x = \frac{5}{3} \pm \frac{2\sqrt{13}}{3} \quad or \quad \frac{5 \pm 2\sqrt{13}}{3}$$