2.7 Complex Numbers

We are very familiar with *real numbers*, those numbers that represent quantities that can be attached to physical meaning like size, amount, account balance, etc. Real numbers can be natural numbers (like the counting numbers 1,2,3,4,…), whole numbers (just add 0 to the natural numbers), integers (positive and negative whole numbers) , rational numbers (fractions containing only integers, like $-\frac{4}{5}$, and decimals that either terminate like $.48$, or repeat, like or $0.353535\ldots = 0.\overline{35}$) and irrational numbers (decimals that do not terminate or repeat, like π). Even the irrational numbers can be thought of with physical meaning. For example, when you think about the circumference of a circle, you will recall the formula $C = 2\pi r$, which means that the distance around a circle is 2 times π times the radius of that circle. Recall that π is just a number that can't be written down numerically without approximating because its decimal expansion goes on forever and never repeats:

$$\pi \approx 3.14159265358979323846264338327 95 \ldots..$$

The best we can do is put "…" at the end, but that is not accurate since it does not repeat. A common approximation for π is 3.14, but if we want to represent the real number, we need a symbol. Another such irrational number that we will encounter is

$$e \approx 2.7182818284590452353602874713527$$

Both of these irrational numbers occur all over the place in nature and finance! They are definitely involved in representing physical or tangible phenomena.

But what other kinds of numbers could there be? And do they represent any kind of tangible phenomena? The short answers are *imaginary numbers* and *yes*.

First, recall that we were unable to take the square root of a negative number due to the fact that any real number times itself will produce a positive number. So, for example, $\sqrt{-1}$ is not a real number because $1^2 = 1 \text{ and } (-1)^2 = 1$. There is no real number we can square and get -1. BUT, we could define a number to be $\sqrt{-1}$ and that is exactly what mathematicians of the past did! We call this number i, the *imaginary unit*. Why would mathematicians do such a thing? Basically, this number was created in order to solve the equation $x^2 + 1 = 0$:

$x^2 + 1 = 0$:

⇨ $x^2 = -1$

⇨ $x = \pm\sqrt{-1}$

⇨ $x = \pm i$

It is worth noting here that we can now factor the sum of squares if we write it as a difference:

$$x^2 + 1 = x^2 - (-1)$$

$$= (x + i)(x - i)$$

since $i = \sqrt{-1}$.

At the time that this imaginary unit was "invented", there really was no tangible application, but later with the rise of quantum mechanics, 'i' found its useful place in our society. In order to understand the quantum physics of current technology, this imaginary unit is imperative. I wonder what our technology would look like today without it…

We can now evaluate the square roots of negative numbers as imaginary numbers:

$$\sqrt{-4} = \sqrt{-1 \cdot 4} = \sqrt{i^2 \cdot 2^2} = 2i$$

An easier way to think of it is to simply 'pull an i out' when you have the square root of a negative number. Then deal with taking the square root of the number as usual.

<u>Examples</u>

Simplify each of the following expressions completely.

1. $\sqrt{-9} = i\sqrt{9} = i\sqrt{3 \cdot 3} = 3i$

2. $\sqrt{-5} = i\sqrt{5}$

   We can't break the 5 down so we can either write our answer as it is above $i\sqrt{5}$ or we could write it as $\sqrt{5}i$ with the number before the i. Just make sure the i is not inside the radical, but outside, whether you place it before or after the real number.

3. $-\sqrt{-8} = -i\sqrt{8} = -i\sqrt{2 \cdot 2 \cdot 2} = -2i\sqrt{2}\ or - 2\sqrt{2}i$

   The negative outside of the radical stays outside, as it is really -1 multiplied by the radical.

4. $\sqrt{-288} = i\sqrt{2 \cdot 144} = 12i\sqrt{2}\ or\ 12\sqrt{2}i$

   We could have broken 144 all the way down, but we didn't really need to since we know $144 = 12 \cdot 12$ from our multiplication table.

5. $\sqrt{-3} \cdot \sqrt{-6} = i\sqrt{3} \cdot i\sqrt{6} = i^2\sqrt{3 \cdot 3 \cdot 2} = -1 \cdot 3\sqrt{2} = -3\sqrt{2}$

   Notice that we dealt with each of these radicals separately before multiplying them together. We could not put them "under the same roof" in the beginning because that rule is only valid for positive radicands. Let's see what would have happened had we incorrectly applied that rule here:

$$\sqrt{-3} \cdot \sqrt{-6} = \sqrt{(-3)(-6)} = \sqrt{3 \cdot 3 \cdot 2} = 3\sqrt{2}$$

We would not have arrived at the correct answer. The rule $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ only applies to positive radicands.

6. $\sqrt{-5} \cdot \sqrt{-35} = i\sqrt{5} \cdot i\sqrt{35} = i^2\sqrt{5 \cdot 5 \cdot 7} = -1 \cdot 5\sqrt{7} = -5\sqrt{7}$

Now that we are familiar with imaginary numbers, we will need to learn how to work with them. Let's consider other integer powers of i. We already know that $i^2 = -1$, but what about i^3, i^4, i^5, etc.? We will discover a pattern that will make computing these powers very simple.

$i^1 = \boldsymbol{i}$

$i^2 = \mathbf{-1}$

$i^3 = i^2 \cdot i = -1 \cdot i = -\boldsymbol{i}$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = \mathbf{1}$

Now watch this pattern repeat (due to the fact that $i^4 = 1$)

$i^5 = i^4 \cdot i = 1 \cdot i = \boldsymbol{i}$

$i^6 = i^4 \cdot i^2 = 1(-1) = \mathbf{-1}$

$i^7 = i^4 \cdot i^3 = 1(-i) = -\boldsymbol{i}$

$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = \mathbf{1}$

$i^9 = i^4 \cdot i^4 \cdot i = 1 \cdot 1 \cdot i = \boldsymbol{i}$

$i^{10} = i^4 \cdot i^4 \cdot i^2 = 1 \cdot 1 \cdot (-1) = \mathbf{-1}$

$i^{11} = i^4 \cdot i^4 \cdot i^3 = 1 \cdot 1 \cdot (-i) = -\boldsymbol{i}$

$i^{12} = i^4 \cdot i^4 \cdot i^4 = 1 \cdot 1 \cdot 1 = \mathbf{1}$

Can you simplify i^{37} by utilizing the fact that $i^4 = 1$? If you use the same approach that we used above (breaking it down into factors of i^4 and evaluate what remains (which must be $i, i^2, or\ i^3$), you can get the answer as follows:

$$i^{37} = i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i = \boldsymbol{i}$$

You will notice that we ended up with nine $i^4{}'s$ above by utilizing the first rule of exponents ($b^m \cdot b^n = b^{m+n}$) to get i^{36}. Another way we could have written this is as follows:

$$i^{37} = i^{36} \cdot i = (i^4)^9 \cdot i = 1^9 \cdot i = \boldsymbol{i}$$

Here we just used the third rule of exponents ($(b^m)^n = b^{mn}$) to break down i^{37}into as many $i^4{}'s$ as we could. This leads us to the simpler approach of dividing the power by 4 and taking the remainder to be the new power. Notice that what really mattered above was how many were left over after we factored out the $i^4{}'s$. Below, this simpler approach is detailed for the same problem:

To evaluate i^{37}, we will divide 37 by 4 and take the remainder as our new exponent:

$$\begin{array}{r} 9 \\ 4\overline{)37} \\ 36 \\ \hline 1 \end{array}$$

This tells you how many $i^4{}'s$ you have.

This tells you how many $i's$ you have leftover.
This is your new exponent!

Therefore,

$$i^{37} = i^1 = i$$

This is a nice fast approach, so we will use it in the next set of examples.

7. i^{12}

Divide 4 into 12 and you will get remainder 0:

$$\begin{array}{r} 3 \\ 4\overline{)12} \\ \underline{12} \\ 0 \end{array}$$

Your new exponent is 0.

So $i^{12} = i^0 = 1$

Note that the long way to do this still gives the same answer:

$i^{12} = i^4 \cdot i^4 \cdot i^4 = 1 \cdot 1 \cdot 1 = 1$

8. i^{27}

$$\begin{array}{r} 6 \\ 4\overline{)27} \\ \underline{24} \\ 3 \end{array}$$

Your new exponent is 3.

So $i^{27} = i^3 = -i$

We know that our remainder must be 0, 1, 2, or 3 when dividing by 4, and for each of these, we know how to simplify: $i^0 = 1,\ \ i^1 = i,$ $i^2 = -1\ and\ i^3 = -i$ as we showed at the beginning of this discussion.

Sometimes, we can use our exponent rules to simplify as well.

9. $(-2i)^5 = (-2)^5 i^5 = -32i$

Here we distributed our exponent over multiplication first and then we evaluated $i^5 = i^1 = i$.

10. $i^9 - (4i)^2 = i - 16i^2 = i - 16(-1) = i + 16\ (or\ 16 + i)$

In the last example, notice that the answer has a real number added to an imaginary number. This is what we call a *complex number*.

Complex Numbers

A **complex number** is a number of the form $a + bi$, where both a and b are real numbers.

Complex numbers are the most general type of numbers that we use. Every number that you have ever dealt with is a complex number. All real numbers are complex numbers and all imaginary numbers are complex numbers. Any number that can be written in the form $a + bi$ is a complex number. The following are examples of complex numbers:

1. $5 + 2i$ (Here $a = 5$ and $b = 2$)

2. $\frac{2}{3} - \frac{1}{8}i$ (Here $a = \frac{2}{3}$ and $b = \frac{1}{8}$)

3. $\sqrt{3} + 2.95i$ (Here $a = \sqrt{3}$ and $b = 2.95$)

4. $7i$ (This can be written as $0 + 7i$, so $a = 0$ and $b = 7$)

5. 8 (This can be written as $8 + 0i$, so $a = 8$ and $b = 0$)

Complex numbers are useful in fields other than pure mathematics, such as electrical engineering and quantum mechanics. We will be working with them from now on and they can occur as solutions to equations from this point forward. In order to fully understand them, we must also be able to perform the same operations on them that we perform on real numbers (addition, subtraction, multiplication, and division). We will focus on these operations for the remainder of this section.

If we want to add or subtract two complex numbers to obtain another complex number, the process is quite simple. You can treat i like a variable and combine "like" terms.

Examples

Perform the indicated operations and simplify. Write your answer in the form $\boldsymbol{a+bi}$.

1. $(3+4i)+(7+9i)=3+4i+7+9i=10+13i$

   All we did was remove the parentheses and combine "like" terms. Notice that the answer is in the form $a+bi$. That is how we know when we are done.

2. $(3+4i)-(7+9i)=3+4i-7-9i=-4-5i$

   For this one, we had to distribute the negative in order to remove the parentheses, but that is really the only difference between addition and subtraction of complex numbers.

In the next few examples, we will multiply. Multiplication of complex numbers is just like multiplication of two binomials. You already know how to FOIL, but you will need to replace any powers of i at the end in order to simplify your answer completely.

3. $(3+4i)(7+9i) = 21 + 27i + 28i + 36i^2$

$$= 21 + 27i + 28i - 36$$

$$= -15 + 55i$$

Notice that we replaced $36i^2$ with -36 because $i^2 = -1$.

4. $(2-8i)(3+5i) = 6 + 10i - 24i - 40i^2$

$$= 6 + 10i - 24i + 40$$

$$= 46 - 14i$$

5. $(7-3i)^2 = (7-3i)(7-3i)$

$$= 49 - 21i - 21i + 9i^2$$

$$= 49 - 21i - 21i - 9$$

$$= 40 - 42i$$

Division is the only operation where the process is different than the process we use with real numbers, but it is still a process we are already familiar with. When we rationalized denominators, our goal was to get rid of the radical in the denominator, and we often utilized the conjugate to do this. When we divide complex numbers, our goal will be to get rid of the i in the denominator, and once again, we will use the conjugate to do this. The conjugate of a complex number $a + bi$ is the complex number $a - bi$. (Basically, you change the sign in the middle just like you did with the radicals). The difference here is that you have to make sure your answer is in the form $a + bi$, so you will need to distribute the denominator to both the real and imaginary parts of the numerator at the end of the problem.

6. $\frac{3+4i}{7+9i} = \frac{3+4i}{7+9i} \cdot \frac{7-9i}{7-9i} = \frac{(3+4i)(7-9i)}{(7+9i)(7-9i)} = \frac{21-27i+28i-36i^2}{130}$

The denominator can be multiplied fairly quickly since you know the middle terms will drop out: $(7+9i)(7-9i) = 49 - 81i^2 = 49 + 81 = 130$

$$= \frac{57 + i}{130}$$

$$= \frac{57}{130} + \frac{1}{130}i$$

The final answer needs to be in the form $a + bi$, so we distributed the denominator to both pieces and we pulled i out of the numerator.

7. $\frac{5+3i}{2-i} = \frac{5+3i}{2-i} \cdot \frac{2+i}{2+i} = \frac{(5+3i)(2+i)}{(2-i)(2+i)} = \frac{10+5i+6i+3i^2}{5}$

$Note$: $(2-i)(2+i) = 4 - i^2 = 4 + 1 = 5$

$$= \frac{7 + 11i}{5}$$

$$= \frac{7}{5} + \frac{11}{5}i$$

8. $\dfrac{7}{i} = \dfrac{7}{i} \cdot \dfrac{-i}{-i} = \dfrac{-7i}{-i^2} = \dfrac{-7i}{1} = -7i$

Notice that the conjugate of i $(= 0 + i)$ is $-i$ $(= 0 - i)$. We could have just multiplied by i, but using the conjugate gets rid of the negative in the bottom as well. Also notice that when either $a = 0$ or $b = 0$, you can omit writing them. (You don't have to write $0 - 7i$).

In the next example, we will go back to simplifying radicals, but now we are allowing a negative inside of the square root. This will help later for simplifying solutions to equations.

9. $\dfrac{30+\sqrt{-125}}{15} = \dfrac{30+i\sqrt{125}}{15} = \dfrac{30+i\sqrt{5\cdot5\cdot5}}{15} = \dfrac{30+5i\sqrt{5}}{15}$

$$= \frac{30}{15} + \frac{5\sqrt{5}}{15}i = 2 + \frac{\sqrt{5}}{3}i$$

Take care of the radical first, as usual, and then reduce your fractions.