2.6 Solving Radical Equations

Now we will turn our attention back to solving equations. Since we know how to operate with radicals now, we should be able to solve basic equations involving them. We already know that a square will "undo" a square root and a cube will "undo" a cube root, etc. We also know that we can operate on any equation as long as we do the same thing to both sides. We can use this knowledge to isolate our variable when it is inside of a root such as this. Please note that this is only allowed with equations, not with expressions. You can't just randomly square or cube an expression because this will change it. You must have two sides of an equation in order to do anything to it (other than rewrite it in a different form).

<u>Examples</u>

Solve each of the following equations for the indicated variable.

1. $\sqrt{x} = 9$

   You probably already can see the answer to this one since you know $\sqrt{81} = 9$, but most radical equations do not have obvious solutions, so we need a technique to solve them. We will demonstrate our technique here, where we already know what the answer should be. In order to isolate the variable, we need to "undo" what has been done to it. We can square both sides to "undo" the square root. The basic technique involves isolating the radical and then squaring both sides.

$$\sqrt{x} = 9$$

The radical is already isolated.

$$\left(\sqrt{x}\right)^2 = 9^2$$
$$x = 81$$

Now square both sides and evaluate.

With radical equations, we must also check our answers by plugging back into the original radical equation:

Is $\sqrt{81} = 9$? Yes, it is!

2. $\sqrt{y-3} = 4$ ⇨ $\left(\sqrt{y-3}\right)^2 = 4^2$

⇨ $y - 3 = 16$

$$\underline{\quad + 3 \quad +3 \quad}$$

⇨ $y = 19$

Now, we must check our answer by plugging it into the original equation: Is $\sqrt{19-3} = 4$? Yes.

3. $\sqrt{5x-9} - 6 = 0$

First, we must isolate the square root. (If we try to square both sides right now, we will not be able to get rid of the radical since $\left(\sqrt{5x-9} - 6\right)^2 = \left(\sqrt{5x-9} - 6\right)\left(\sqrt{5x-9} - 6\right)$ and you would need to FOIL this out to obtain a result that still has a radical in it! Remember that exponents do not distribute over addition or subtraction, as tempting as it may be….)

$$\sqrt{5x-9} - 6 = 0$$
$$\quad\quad +6 \quad +6$$

Isolate the radical first by adding 6 to both sides.

⇨ $\sqrt{5x-9} = 6$

⇨ $\left(\sqrt{5x-9}\right)^2 = 6^2$

Now square both sides.

⇨ $5x - 9 = 36$
$$+9 \quad +9$$

⇨ $5x = 45$

⇨ $x = 9$

Check: $\sqrt{5 \cdot 9 - 9} - 6 = 0$
$$\sqrt{45-9} - 6 = 0$$
$$\sqrt{36} - 6 = 0$$
$$6 - 6 = 0 \checkmark$$

Sometimes, we will get more than one answer. We must check ALL answers by plugging each into the original equation. If a solution doesn't satisfy the original equation, it is called *extraneous*. The next example has an extraneous solution.

4. $\sqrt{z+2} = z$ ⇨ $\left(\sqrt{z+2}\right)^2 = z^2$

The radical is already isolated, so we can go ahead and square both sides.

⇨ $z + 2 = z^2$
$$-z-2 \quad\quad -z-2$$

⇨ $0 = z^2 - z - 2$

We end up with a quadratic, so we need to get everything on one side to set it equal to 0.

➪ $0 = (z-2)(z+1)$

➪ $z-2=0 \quad ; \; z+1=0$

➪ $z=2\,; \qquad z=-1$

Now factor and set each factor equal to 0 to obtain two possible solutions.

Check each answer:

$z=2$

$\sqrt{2+2}=2$

$\sqrt{4}=2$ ✓

$z=-1$

$\sqrt{-1+2}=-1$

$\sqrt{1}=-1$ Not true

Therefore, the only true solution to the original equation is $z=2$. The other solution is extraneous.

5. $\sqrt{x+1}+5=x$ ➪ $\sqrt{x+1}+5=x$

$$\begin{array}{r} \sqrt{x+1}+5 = x \\ -5 \quad\; -5 \\ \hline \end{array}$$

Isolate the radical first by subtracting 5 from both sides.

➪ $\left(\sqrt{x+1}\right)^2 = (x-5)^2$

Now square both sides to eliminate the radical. You must FOIL here.

➪ $x+1 = x^2-10x+25$

$$\begin{array}{r} -x-1 \qquad\qquad -x \quad -1 \\ \hline \end{array}$$

➪ $0 = x^2-11x+24$

This is a quadratic, so we need to get everything on one side to set it equal to 0.

➪ $0=(x-8)(x-3)$

➪ $x-8=0 \quad ; \; x-3=0$

➪ $x=8\,; \qquad x=3$

Now factor and set each factor equal to 0 to obtain two possible solutions.

Check each answer:

$x = 8$ | $x = 3$

$\sqrt{8+1} + 5 = 8$ | $\sqrt{3+1} + 5 = 3$

$\sqrt{9} + 5 = 8$ | $\sqrt{4} + 5 = 3$ Not true

$3 + 5 = 8$ ✓

Therefore, the only true solution to the original equation is $x = 8$. The other solution is extraneous.

6. $\sqrt{28 - y} - 2 = y$ ⇨

$$\begin{array}{r} \sqrt{28-y} - 2 = y \\ +2 \quad\quad +2 \\ \hline \end{array}$$

Isolate the radical first by adding 2 to both sides.

⇨ $\left(\sqrt{28-y}\right)^2 = (y+2)^2$

⇨ $28 - y = y^2 + 4y + 4$

Now square both sides to eliminate the radical. You must FOIL here.

$$\begin{array}{r} -28 + y \quad\quad + y \ - 28 \\ \hline \end{array}$$

⇨ $0 = y^2 + 5y - 24$

This is a quadratic, so we need to get everything on one side to set it equal to 0.

⇨ $0 = (y + 8)(y - 3)$

⇨ $y + 8 = 0 \;\; ; \; y - 3 = 0$

⇨ $y = -8 \, ; \quad y = 3$

Now factor and set each factor equal to 0 to obtain two possible solutions.

Check each answer:

$$y = -8$$
$$\sqrt{28-(-8)} - 2 = -8$$
$$\sqrt{36} - 2 = -8$$
$$6 - 2 = -8 \quad \text{Not true}$$

$$y = 3$$
$$\sqrt{28-3} - 2 = 3$$
$$\sqrt{25} - 2 = 3$$
$$5 - 2 = 3 \checkmark$$

Therefore, the only true solution to the original equation is $y = 3$. The other solution is extraneous.

The next two examples have more than one radical in the equation and will require repeating the process of isolating a radical and squaring both sides.

7. $\sqrt{x+4} + \sqrt{x-1} = 5$

What would happen if we squared both sides right now? This would be a bit messy since we cannot distribute exponents over the addition and we would need to multiply out the left side as follows:

$$\left(\sqrt{x+4} + \sqrt{x-1}\right)^2 = \left(\sqrt{x+4} + \sqrt{x-1}\right)\left(\sqrt{x+4} + \sqrt{x-1}\right)$$
$$= x - 4 + 2\sqrt{x+4} \cdot \sqrt{x-1} + x - 1 = 2x + 2\sqrt{(x+4)(x-1)} - 5$$

This is a messier radical than the ones we started with and although we could proceed with this method, there is a better way. By eliminating one radical at a time, we will be able to keep the radicands simpler and the numbers smaller. It is simpler to isolate one of the radicals and square both sides and then isolate the remaining radical and square both sides again. It doesn't matter which radical you choose to isolate. You will get the same answer in the end.

$$\sqrt{x+4}+\sqrt{x-1}=5$$

Isolate either radical first by taking the other one to the other side.

⇨ $\sqrt{x+4}=5-\sqrt{x-1}$

Now square both sides to eliminate one of the radicals. You must FOIL here.

⇨ $\left(\sqrt{x+4}\right)^2=\left(5-\sqrt{x-1}\right)^2$

⇨ $x+4=25-10\sqrt{x-1}+x-1$

Clean up the right side a bit and then isolate the remaining radical.

⇨ $x+4 \;=\; 24-10\sqrt{x-1}\;+x$

$\underline{-x-24 \quad -24 \qquad\qquad\qquad -x}$

⇨ $-20=-10\sqrt{x-1}$

You can square both sides without isolating the radical *completely* if you like (since exponents **do** distribute over multiplication), but we chose to go ahead and divide by -10 to isolate the radical completely.

⇨ $\dfrac{-20}{-10}=\dfrac{-10\sqrt{x-1}}{-10}$

⇨ $2=\sqrt{x-1}$

Now square both sides again to eliminate the remaining radical.

⇨ $2^2=\left(\sqrt{x-1}\right)^2$

⇨ $4 \;=x-1$

⇨ $5=x$

Check: $x=5$ $\quad \sqrt{5+4}+\sqrt{5-1}=5$

$$\sqrt{9}+\sqrt{4}=5$$

$$3+2=5$$ ✔

8. $\sqrt{t-1} + \sqrt{t+15} = 4$

⇨ $\sqrt{t-1} = 4 - \sqrt{t+15}$

⇨ $\left(\sqrt{t-1}\right)^2 = \left(4 - \sqrt{t+15}\right)^2$

⇨ $t - 1 = 16 - 8\sqrt{t+15} + t + 15$

⇨ $t - 1 \;=\; 31 - 8\sqrt{t+15} \quad + t$

$\underline{-t - 31 \quad - 31 \qquad\qquad\qquad\qquad - t}$

⇨ $-32 = -8\sqrt{t+15}$

⇨ $\dfrac{-32}{-8} = \dfrac{-8\sqrt{t+15}}{-8}$

⇨ $4 = \sqrt{t+15}$

⇨ $4^2 = \left(\sqrt{t+15}\right)^2$

⇨ $16 \;= t + 15$

⇨ $1 = t$

Check: $t = 1$

$$\sqrt{1-1} + \sqrt{1+15} = 4$$

$$\sqrt{0} + \sqrt{16} = 4$$

$$0 + 4 = 4$$

We will now turn our attention to equations involving other roots and rational exponents. The idea is the same, except the "undoing" of the root will require a different power to be applied to both sides other than the square.

9. $\sqrt[3]{7n-1} = 3$

   To "undo" the cube root, we will simply cube both sides:

$$\left(\sqrt[3]{7n-1}\right)^3 = 3^3$$

$$\Rightarrow \quad 7n - 1 = 27$$

$$\Rightarrow \quad 7n = 28$$

$$\Rightarrow \quad n = 4$$

   We should check our answers for all radical equations, even though we know that cube roots (and other odd roots) do not typically have any issues with extraneous answers. If you plug 4 into the original equation, you will see that it makes the equation true.

You may notice that the next example is once again a cube root, but it has been written in a different way. You can either write it as a cube root and cube both sides or use a more general method that will work for any equation with a single rational exponent. In order to get rid of a rational exponent, we can simply raise both sides to the reciprocal power, as that will make the power 1 when we use the third exponent rule to multiply the powers together: $(b^m)^n = b^{mn}$.

10. $(x^3 - 7)^{\frac{1}{3}} = x - 1$

Cube both sides since 3 is the reciprocal of $\frac{1}{3}$. Thus, $\frac{1}{3} \cdot 3 = 1$ and that will give us $(x^3 - 7)^1$ getting rid of the rational exponent.

⇨ $\left((x^3 - 7)^{\frac{1}{3}}\right)^3 = (x - 1)^3$

⇨ $x^3 - 7 = (x - 1)(x - 1)(x - 1)$

Don't forget what $(x - 1)^3$ means. You can't distribute the exponent over subtraction. You must FOIL this out, multiplying two of the factors together and then your result by the third.

⇨ $x^3 - 7 = x^3 - 3x^2 + 3x - 1$

⇨
$$\begin{array}{rl} x^3 - 7 &= x^3 - 3x^2 + 3x - 1 \\ -x^3 + 7 & -x^3 \qquad\qquad\quad + 7 \\ \hline 0 &= -3x^2 + 3x + 6 \end{array}$$

When factoring, don't forget to pull out the GCF first, and if the leading term is negative, pull out the negative to make factoring easier. Do one step at a time.

⇨ $0 = -3(x^2 - x - 2)$

⇨ $0 = -3(x - 2)(x + 1)$

⇨ $x - 2 = 0 \quad ; \quad x + 1 = 0$

Reminder: There is no need to set the constant factor equal to 0 . Only the factors with variables should be set equal to 0.

⇨ $x = 2 \quad ; \quad x = -1$

Now check both answers:

$x = 2$

$(2^3 - 7)^{\frac{1}{3}} = 2 - 1$

$(8 - 7)^{\frac{1}{3}} = 1$

$(1)^{\frac{1}{3}} = 1$ ✔

$x = -1$

$((-1)^3 - 7)^{\frac{1}{3}} = (-1) - 1$

$(-1 - 7)^{\frac{1}{3}} = -2$

$(-8)^{\frac{1}{3}} = -2$ ✔

Both answers are solutions!

11. $x^{\frac{2}{3}} = 2 \quad ➪ \quad \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \pm 2^{\frac{3}{2}}$

Here, we put both sides to the power $\frac{3}{2}$ since it is the reciprocal of $\frac{2}{3}$ and thus $\frac{2}{3} \cdot \frac{3}{2} = 1$, so we get x^1 $(or\ x)$. Don't forget that you need a $\pm$ symbol since you are square rooting both sides with the 2 in the denominator of the power.

$➪ \quad x = \pm 2^{\frac{3}{2}}$

$➪ \quad x = \pm\sqrt{2^3} = \pm 2\sqrt{2}$ (in radical form)

Checking is obvious here if you plug it back in while it is in the rational exponent form $2^{\frac{3}{2}}$.

12. $3 + x^{-\frac{5}{2}} = 35$

First, we must isolate $x^{-\frac{5}{2}}$ in order to "undo" the power:

$$\begin{array}{r} 3 + x^{-\frac{5}{2}} = 35 \\ -3 \qquad\qquad -3 \\ \hline x^{-\frac{5}{2}} = 32 \end{array}$$

Now, put both sides to the reciprocal power:

$$\left(\quad x^{-\frac{5}{2}}\right)^{-\frac{2}{5}} = (32)^{-\frac{2}{5}}$$

$$➪ \quad x = \frac{1}{(32)^{\frac{2}{5}}} = \frac{1}{\left(\sqrt[5]{32}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$$

Now check your answer:

$$3 + \left(\frac{1}{4}\right)^{-\frac{5}{2}} = 35$$

$$3 + (4)^{\frac{5}{2}} = 35$$

$$3 + \left(\sqrt{4}\right)^{5} = 35$$

$$3 + (2)^{5} = 35$$

$$3 + 32 = 35 \checkmark$$

Sometimes we have many variables and wish to solve for one of them, as in the next example, but the technique remains the same.

13. $t = \sqrt[5]{\frac{L}{m}} + 7$ Solve for m

$$t = \sqrt[5]{\frac{L}{m}} + 7$$

⇨ $$t - 7 = \sqrt[5]{\frac{L}{m}}$$

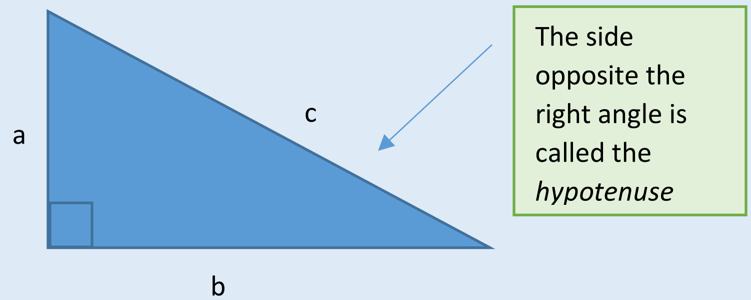
⇨ $$(t - 7)^5 = \left(\sqrt[5]{\frac{L}{m}}\right)^5$$

⇨ $$(t - 7)^5 = \frac{L}{m}$$

⇨ $$m = \frac{L}{(t-7)^5}$$

<u>Pythagorean Theorem</u>

Given any *right* triangle (a triangle that contains a 90° angle), the lengths of the three sides of the triangle are related by the following equation: $a^2 + b^2 = c^2$

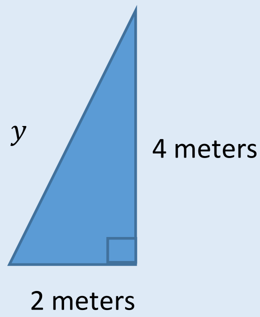


This theorem is fundamental in trigonometry and very useful for some engineering applications. We are studying this theorem here because this equation involves using radicals to solve for a given side. You will encounter use in your future math (and probably science) courses.

<u>Examples</u>

Find the length of the missing side for each of the following triangle:

1.



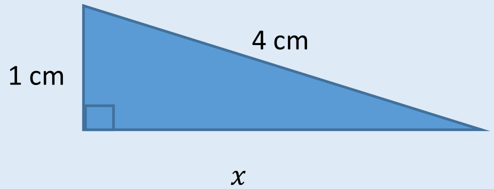
In this example, we are looking for the hypotenuse, y, which is alone on one side of the equation. It is important to note whether you are solving for the hypotenuse or for one of the legs in order to know where your variable goes in the equation.

$$4^2 + 2^2 = y^2$$
$$\Rightarrow \quad 16 + 4 = y^2$$
$$\Rightarrow \quad 20 = y^2$$
$$\Rightarrow \quad y = \pm\sqrt{20} = \pm 2\sqrt{5}$$
$$\Rightarrow \quad y = 2\sqrt{5}\ meters$$

Even though we technically get two roots to our mathematical equation, we choose the positive one since our answer represents length, which is always positive.

(or $y = 4.47\ meters$)

2.



Notice that we are solving for one of the legs in this example, not the hypotenuse.

$$1^2 + x^2 = 4^2$$
$$\Rightarrow \quad 1 + x^2 = 16$$
$$\Rightarrow \quad x^2 = 15$$
$$\Rightarrow \quad x = \pm\sqrt{15}$$
$$\Rightarrow \quad x = \sqrt{15}\ cm$$

or $x = 3.87\ cm$