2.5 Radical Expressions: Rationalizing the Denominator

Rationalizing the denominator just means writing the expression without a radical in the denominator. We can always do this because we can rewrite any fraction by multiplying the numerator and denominator by the same thing (in essence multiplying by 1). But why would we want to do this? Well, it is much easier to divide by a whole number than by a radical, both on a calculator and in our minds when we try to compute estimates. In your future math courses, your answers will usually be given this way, so we address it here.

The way to think about it is this: What can we multiply the denominator by that will cause the denominator to become a whole number? What will get rid of the radical? Whatever it is, we must multiply the numerator by it as well so that we do not change our fraction. We are simply rewriting it in a different form.

Given the fraction $\frac{1}{\sqrt{7}}$, what can we multiply the denominator by so that the radical disappears? You are probably thinking $\sqrt{7}$, since we know $\sqrt{7} \cdot \sqrt{7} = 7$. This is correct. But make sure that you also multiply the numerator by $\sqrt{7}$, so that you are really multiplying your fraction by $\frac{\sqrt{7}}{\sqrt{7}}$, which is just equal to 1. Multiplying by 1 doesn't change the value of the original fraction. We are only changing the form in which the fraction is written:

$$\frac{1}{\sqrt{7}} = \frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{7}$$

<u>Examples</u>

Simplify each of the following radical expressions by rationalizing the denominator. Assume all variables represent non-negative real numbers.

1. $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

   Note: This number will occur quite a bit in trigonometry and you will see it in both forms there.

2. $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

3. $\frac{8\sqrt{5}}{\sqrt{2}} = \frac{8\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{10}}{2} = 4\sqrt{10}$

   We should always simplify our answers as much as is possible.

4. $\sqrt{\frac{7}{3}} = \frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{3}$

   When the radical is over the entire fraction, it is useful to distribute it to the numerator and denominator separately before rationalizing since you want to be able to look at the denominator separately and decide what you need to multiply by.

5. $\frac{3+\sqrt{5}}{\sqrt{7}} = \frac{3+\sqrt{5}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{(3+\sqrt{5})\sqrt{7}}{7} = \frac{3\sqrt{7}+\sqrt{35}}{7}$

Here we have to remember that when we multiply the numerators together, the $\sqrt{7}$ needs to be distributed to both terms in the other numerator. If it helps, place parentheses around the other numerator like we did here. Also, be careful not to try to cancel the 7 on the bottom with the inside of either radical in the top. It is a common error.

6. $\frac{4\sqrt{2}-\sqrt{3}}{\sqrt{5}} = \frac{4\sqrt{2}-\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{(4\sqrt{2}-\sqrt{3})\sqrt{5}}{5} = \frac{4\sqrt{10}-\sqrt{15}}{5}$

The examples we have seen so far had a denominator with a single radical and so we knew we had to multiply by that same radical in order to eliminate it. But, denominators can be more complicated. What if there is a sum or difference in the denominator? Just multiplying by the radical won't get rid of it…We need to be a bit more creative. Consider the following example:

7. $\frac{6}{7+\sqrt{5}}$

If we try multiplying that denominator by $\sqrt{5}$, what would we get? Would the radical disappear? Try it and you will see that it will not. If we multiplied the denominator by $\sqrt{5}$, we would obtain $7\sqrt{5}+5$. This does not help us at all! There is a way to get rid of that radical, though. If we multiply the denominator by its *conjugate*[1] $7-\sqrt{5}$, then watch what happens:

[1] (To get the *conjugate*, just take the same number with the opposite sign on the radical. We will use the conjugate again in section 2.7 to get rid of imaginary numbers in denominators.)

$$= \frac{6}{7+\sqrt{5}} \cdot \frac{7-\sqrt{5}}{7-\sqrt{5}} = \frac{6(7-\sqrt{5})}{(7+\sqrt{5})(7-\sqrt{5})} = \frac{42-6\sqrt{5}}{44}$$

Here, you can quickly compute the denominator since the middle terms will always drop out, taking the radical with them!
$(7+\sqrt{5})(7-\sqrt{5}) = 49 - 7\sqrt{5} + 7\sqrt{5} - 5 = 49 - 5 = 44$
The shortcut is to just multiply the front and back and subtract, since you know the middle terms will drop out. This takes you to $49 - 5$ very quickly! But we must still simplify our fraction when we can…

$$= \frac{2(21-3\sqrt{5})}{44} = \frac{21-3\sqrt{5}}{22}$$

8. $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$

$$= \frac{5+\sqrt{10}+\sqrt{10}+2}{5-2} = \frac{7+2\sqrt{10}}{3}$$

9. $\frac{2\sqrt{x}+5}{3\sqrt{x}-\sqrt{y}} = \frac{2\sqrt{x}+5}{3\sqrt{x}-\sqrt{y}} \cdot \frac{3\sqrt{x}+\sqrt{y}}{3\sqrt{x}+\sqrt{y}} = \frac{6x+2\sqrt{xy}+15\sqrt{x}+5\sqrt{y}}{9x-y}$

10. $\frac{3+5\sqrt{a}}{2\sqrt{a}+7} = \frac{3+5\sqrt{a}}{2\sqrt{a}+7} \cdot \frac{2\sqrt{a}-7}{2\sqrt{a}-7} = \frac{6\sqrt{a}-21+10a-35\sqrt{a}}{4a-49}$

$$= \frac{10a - 29\sqrt{a} - 21}{4a - 49}$$

What happens if we have a different index in the denominator? We know that in order to eliminate a radical, we need to use the exchange rate (or index) to decide how many more of something we need to multiply by to get a whole number. For example, if we have a cube root, we will need three of them, but if we have a fourth root then we will need four of them, etc…

11. $$\frac{2}{\sqrt[3]{6}} = \frac{2}{\sqrt[3]{6}} \cdot \frac{\sqrt[3]{6}}{\sqrt[3]{6}} \cdot \frac{\sqrt[3]{6}}{\sqrt[3]{6}} = \frac{2\sqrt[3]{36}}{6} = \frac{\sqrt[3]{36}}{3}$$

You might notice that you could write this in a different way by realizing you need two more 6's to give you three of them in the denominator and then just multiply by $\sqrt[3]{6^2}$ in both the denominator and the numerator. Either way, you will get the same answer.

12. $$\sqrt[3]{\frac{5}{7}} = \frac{\sqrt[3]{5}}{\sqrt[3]{7}} = \frac{\sqrt[3]{5}}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7}}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7}}{\sqrt[3]{7}} \left(or \ = \frac{\sqrt[3]{5}}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}} \right) = \frac{\sqrt[3]{245}}{7}$$

13. $$\frac{\sqrt[3]{9x}}{\sqrt[3]{3xy}} = \sqrt[3]{\frac{9x}{3xy}} = \sqrt[3]{\frac{3}{y}} = \frac{\sqrt[3]{3}}{\sqrt[3]{y}}$$

Here we noticed that we could simplify before rationalizing, which is always a good idea. Try this problem without simplifying first and you will see you end up simplifying at the end, but then you get the same answer.

$$= \frac{\sqrt[3]{3}}{\sqrt[3]{y}} \cdot \frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} = \frac{\sqrt[3]{3y^2}}{y}$$

14. $$\sqrt[4]{\frac{16}{9y^3}} = \frac{\sqrt[4]{16}}{\sqrt[4]{9y^3}} = \frac{2}{\sqrt[4]{9y^3}} = \frac{2}{\sqrt[4]{9y^3}} \cdot \frac{\sqrt[4]{9y}}{\sqrt[4]{9y}} = \frac{2\sqrt[4]{9y}}{3y}$$

Notice that in this denominator, we already had two 3's and three y's under the fourth root because $\sqrt[4]{9y^3} = \sqrt[4]{3 \cdot 3 \cdot y \cdot y \cdot y}$, so we only needed two more 3's and one more y, which means we only need to multiply by $\sqrt[4]{3 \cdot 3 \cdot y}$ or $\sqrt[4]{9y}$.

15. $$\sqrt[5]{\frac{32}{81x^3}} = \frac{\sqrt[5]{32}}{\sqrt[5]{81x^3}} = \frac{2}{\sqrt[5]{81x^3}} = \frac{2}{\sqrt[5]{81x^3}} \cdot \frac{\sqrt[5]{3x^2}}{\sqrt[5]{3x^2}} = \frac{2\sqrt[5]{3x^2}}{3x}$$

Notice that in this denominator, we already had four 3's and three x's under the fifth root because $\sqrt[5]{81x^3} = \sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x}$, so we only needed one more 3 and two more x's, which means we only need to multiply by $\sqrt[5]{3 \cdot x \cdot x}$ or $\sqrt[5]{3x^2}$.