2.4 Adding and Subtracting Radical Expressions

We already know how to count things by combining "like" terms. This is the same skill we will use when adding and subtracting radicals. As a matter of fact, it is the same skill we use when adding or subtracting anything in mathematics. You will use this skill over and over again in your math courses. For example, if we have $2\ x's$ and we add 9 more $x's$, then we should have a totals of $11\ x's$. This can be represented by the following mathematical equation: $2x + 9x = 11x$

Notice that we must be counting the same type of thing in order to count them together, however. If we instead had $2\ x's$ and $9\ y's$, represented by the expression $2x + 9y$, we would not be able to combine them into a single term because we do not have 11 of the same type of thing. This is true with radicals as well:

$$2\sqrt{3} + 9\sqrt{3} = 11\sqrt{3}$$

But $2\sqrt{3} + 9\sqrt{7}$ cannot be combined.

Another way that you could think about this involves factoring. We can see that we could factor the expression $2\sqrt{3} + 9\sqrt{3}$ by factoring out $\sqrt{3}$. This would give us $\sqrt{3}(2 + 9) = \sqrt{3} \cdot 11\ or\ 11\sqrt{3}$. This is not the case with $2\sqrt{3} + 9\sqrt{7}$, however. There is no common factor here.

Reminder: We must pay attention to the operation because addition and multiplication require different procedures. For example, if we instead had $2\sqrt{3} \cdot 9\sqrt{3}$, we would get $18 \cdot 3 = 54$ and if we had $\left(2\sqrt{3}\right)\left(9\sqrt{7}\right)$, we would indeed be able to combine them to get $18\sqrt{21}$.

<u>Examples</u>

Perform the indicated operations and simplify. Assume that all variables represent non-negative real numbers.

1. $4\sqrt{5} + 3\sqrt{5} = 7\sqrt{5}$

   What if this were multiplication instead? What would $\left(4\sqrt{5}\right)\left(3\sqrt{5}\right)$ be? Try it and see how it is different.

2. $8\sqrt{7} - 2\sqrt{7} = 6\sqrt{7}$

   Subtraction is also counting, so we combine "like" terms just as we do when we are adding. You can also think of subtraction as adding a negative, so it makes sense that the process is the same.

3. $7\sqrt{10} + 8\sqrt{5}$

   This example is already simplified as the radicals do no match. There is nothing to do here.

4. $2\sqrt{10} - 7\sqrt{40}$

   While at first glance, it appears that these cannot be combined, if we simplify our radicals completely first, then we will see that we can combine our results as follows:

$$2\sqrt{10} - 7\sqrt{40} = 2\sqrt{10} - 7\sqrt{2 \cdot 2 \cdot 10}$$

$$= 2\sqrt{10} - 7 \cdot 2\sqrt{10}$$

$$= 2\sqrt{10} - 14\sqrt{10}$$

$$= -12\sqrt{10}$$

5. $2\sqrt[3]{16} - \sqrt[3]{54} + 3\sqrt[3]{128}$

Once again, we need to break down our radicands and pull stuff out to simplify each one before we can decide what can be combined here. You will need to do a factor tree for each radicand.

$= 2\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2} - \sqrt[3]{2 \cdot 3 \cdot 3 \cdot 3} + 3\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

$= 2 \cdot 2\sqrt[3]{2} - 3\sqrt[3]{2} + 3 \cdot 2 \cdot 2\sqrt[3]{2}$

$= 4\sqrt[3]{2} - 3\sqrt[3]{2} + 12\sqrt[3]{2}$

$= 13\sqrt[3]{2}$

6. $3\sqrt[4]{19} + 4\sqrt[4]{19} - \sqrt{19}$

Notice that not all of the indexes match here. The indexes also need to match or they are not “like” terms, so we can only combine the first two.

$= 7\sqrt[4]{19} - \sqrt{19}$

7. $7\sqrt{5x^3} + 3x\sqrt{125x}$

$= 7x\sqrt{5x} + 3x\sqrt{5 \cdot 5 \cdot 5x}$

$= 7x\sqrt{5x} + 15x\sqrt{5x} = 22x\sqrt{5x}$

8. $\sqrt[5]{x^6y^2} + \sqrt[5]{32x^6y^2} + \sqrt[5]{x^6y^7}$

$= x\sqrt[5]{xy^2} + 2x\sqrt[5]{xy^2} + xy\sqrt[5]{xy^2}$

$= 3x\sqrt[5]{xy^2} + xy\sqrt[5]{xy^2}$

(or $(3+y)x\sqrt[5]{xy^2}$ if we factor out $x\sqrt[5]{xy^2}$)

9. $\sqrt{9x+9} - \sqrt{4x+4}$

Be careful not to distribute the radical over addition here. We must factor the radicands before we can simplify….

$= \sqrt{9(x+1)} - \sqrt{4(x+1)}$

$= \sqrt{3 \cdot 3(x+1)} - \sqrt{2 \cdot 2(x+1)}$

$= 3\sqrt{x+1} - 2\sqrt{x+1}$

$= \sqrt{x+1}$

10. $\sqrt{32x^2 + 48xy + 18y^2} + \sqrt{2x^2}$

Make sure you begin by factoring the radicands when possible….

$$= \sqrt{2(16x^2 + 24xy + 9y^2)} + x\sqrt{2}$$

$$= \sqrt{2(4x + 3y)^2} + x\sqrt{2}$$

$$= (4x + 3y)\sqrt{2} + x\sqrt{2}$$

$= (4x + 3y + x)\sqrt{2}$ if we factor out $\sqrt{2}$

$$= (5x + 3y)\sqrt{2}$$

We are now ready to put the concepts of addition and multiplication together in the same problems. This combination occurs any time we need to distribute multiplication over addition/subtraction.

11. $\sqrt{3}\left(\sqrt{5} + \sqrt{2}\right) = \sqrt{3} \cdot \sqrt{5} + \sqrt{3} \cdot \sqrt{2} = \sqrt{15} + \sqrt{6}$

As you can see, the idea is the same with radicals as it is with any number or variable. The only difference is that we need to know how to multiply and add radicals to simplify our results.

12. $\sqrt{7}(5+\sqrt{7}) = 5\sqrt{7} + \sqrt{7}\cdot\sqrt{7} = 5\sqrt{7} + 7$

Note that 5 is a whole number – it is not inside of a radical. It must stay outside when we multiply by $\sqrt{7}$.

13. $(1-\sqrt{2})(4+3\sqrt{5}) = 1\cdot 4 + 1\cdot 3\sqrt{5} - 4\sqrt{2} - 3\sqrt{5}\cdot\sqrt{2}$

$= 4 + 3\sqrt{5} - 4\sqrt{2} - 3\sqrt{10}$

Here we have an example where we must FOIL just as we have done with binomials in the past.

14. $(2\sqrt{3}+7)(3\sqrt{5}-\sqrt{2})$

$= 2\sqrt{3}\cdot 3\sqrt{5} - 2\sqrt{3}\cdot\sqrt{2} + 7\cdot 3\sqrt{5} - 7\sqrt{2}$

$= 6\sqrt{15} - 2\sqrt{6} + 21\sqrt{5} - 7\sqrt{2}$

Once again, we must FOIL, or distribute each term in the first factor to each term in the second factor. Just be careful to multiply outsides and insides of radicals correctly and to combine "like" terms at the end if you have any.

For the next example, we need to remember that exponents do not distribute over addition/subtraction and we can either write out what it means or we can use a binomial square formula (square the front, square the back and double the product of them).

15. $\left(3\sqrt{5}-2\right)^2 = \left(3\sqrt{5}-2\right)\left(3\sqrt{5}-2\right)$

$$= 3\sqrt{5}\cdot 3\sqrt{5} - 2\cdot 3\sqrt{5} - 2\cdot 3\sqrt{5} + 2\cdot 2$$

$$= 9\cdot 5 - 6\sqrt{5} - 6\sqrt{5} + 4$$

$$= 45 - 12\sqrt{5} + 4$$

$$= 49 - 12\sqrt{5}$$

16. $\left(\sqrt[3]{x}+5\right)\left(\sqrt[3]{x}-7\right) = \sqrt[3]{x^2} - 7\sqrt[3]{x} + 5\sqrt[3]{x} - 35$

$$= \sqrt[3]{x^2} - 2\sqrt[3]{x} - 35$$

The answer could also be written with exponents rather than radicals: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 35$. Generally, however, we will write our answers in the same format in which the problem was given. It is just good to keep in mind these connections as we learn.

17. $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$

$$= \sqrt{a}\cdot\sqrt{a} - \sqrt{a}\cdot\sqrt{b} + \sqrt{a}\cdot\sqrt{b} - \sqrt{b}\cdot\sqrt{b}$$

$$= a - b$$

This example should remind you of the difference of two squares formula. Notice that the middle terms drop out.

The next example has three terms in the second factor, but we are still simply distributing each term in the first factor to each term in the second factor.

18. $\left(5\sqrt{x}+\sqrt{y}\right)\left(3\sqrt{x}-4\sqrt{y}+7y\right)$

$= 15x - 20\sqrt{xy} + 35y\sqrt{x} + 3\sqrt{xy} - 4y + 7y\sqrt{y}$

$= 15x - 17\sqrt{xy} + 35y\sqrt{x} - 4y + 7y\sqrt{y}$

At this point, the connections should be clear between past material and the last two sections and we should be able to multiply the radicals rather easily. For example, multiplying $5\sqrt{x} \cdot 3\sqrt{x}$ above can be quickly seen as $15x$.

19. $\left(\sqrt{x-7}-5\right)^2$

$= x - 7 - 10\sqrt{x-7} + 25$

$= x + 18 - 10\sqrt{x-7}$

This example was done using the binomial square formula, but you could easily FOIL it out as well to obtain the same result.

We will now turn our attention to simplifying certain kinds of expressions involving radicals. We are doing this because very soon we will be working with the quadratic formula and often our answers have radicals in them and they need to be simplified. This seems like a good place to practice this so that later on it will not be a stumbling block when you have other things to learn.

Simplifying Quotients involving Radicals

20. $$\frac{15+5\sqrt{2}}{25} = \frac{5\left(3+\sqrt{2}\right)}{25} = \frac{\cancel{5}\left(3+\sqrt{2}\right)}{\cancel{5}\cdot 5} = \frac{3+\sqrt{2}}{5}$$

One way to think of simplifying here is to factor and cancel. It is not the only way, of course. You could also distribute the denominator to both pieces of the numerator and simplify the two fraction separately if you like: $\frac{15}{25} + \frac{5\sqrt{2}}{25} = \frac{3}{5} + \frac{\sqrt{2}}{5}$. This means the same thing. You could also simply divide all three of them $(15, 5\sqrt{2}, and\ 25)$ by the same number as a shortcut, but make sure you do it to ALL of them since you cannot cancel over addition/subtraction.

21. $$\frac{-27+9\sqrt{13}}{36} = \frac{9\left(-3+\sqrt{13}\right)}{36} = \frac{\cancel{9}\left(-3+\sqrt{13}\right)}{\cancel{9}\cdot 4} = \frac{-3+\sqrt{13}}{4}$$

Sometimes, you will need to simplify the radical before you reduce the fraction. Do this whenever it is possible to do, just as we do in the next example:

22. $$\frac{14+5\sqrt{45}}{10} = \frac{14+5\sqrt{3\cdot 3\cdot 5}}{10} = \frac{14+15\sqrt{5}}{10}$$

After we have simplified the radical, we will check to see if the numerator has a common factor that will cancel with the denominator. Here, we do not, but we could simplify the fractions separately as follows: $\frac{14+15\sqrt{5}}{10} = \frac{14}{10} + \frac{15\sqrt{5}}{10} = \frac{7}{2} + \frac{3\sqrt{5}}{2}$
This answer is equivalent to the answer above and they are both acceptable.

23. $$\frac{18+12\sqrt{27}}{24} = \frac{18+12\sqrt{3\cdot3\cdot3}}{24} = \frac{18+36\sqrt{3}}{24} = \frac{18\left(1+2\sqrt{3}\right)}{24}$$

$$= \frac{\cancel{6}\cdot3\left(1+2\sqrt{3}\right)}{\cancel{6}\cdot4} = \frac{3\left(1+2\sqrt{3}\right)}{4}$$

You can also write your answer as $\frac{3}{4}\left(1+2\sqrt{3}\right)$ or you can distribute the 3 in the numerator or you can separate it into two fractions. You have the freedom to write your answer in so many forms!