2.3 Multiplying and Dividing Radical Expressions

Within the next two sections, we will explore the differences between the processes of addition/subtraction and multiplication/division involving radicals. These processes are sometimes confused, so it is best to go back to the basic concepts of addition and multiplication whenever confusion arises. Consider the following two computations with the same numbers but with different operations between them:

$$3\sqrt{2} + 3\sqrt{2} \qquad\qquad 3\sqrt{2} \cdot 3\sqrt{2}$$

Can you predict the outcomes for each of these operations? You actually already have the tools to do both of these problems, but these tools have not yet been put in the context of radicals.

Consider first the addition problem. What if the problem was given with a variable representing $\sqrt{2}$? For example, if you had instead $3x + 3x$, where $x = \sqrt{2}$. You already know how to add this because you know adding is just counting. If you have 3 $x's$ to begin with and you add another 3 $x's$, you end up with 6 $x's$. So we know that $3x + 3x = 6x$. In the same way, you can imagine counting $\sqrt{2}'s$. So $3\sqrt{2} + 3\sqrt{2}$ should equal $6\sqrt{2}$. We will discuss this more in the next section.

Let's turn our attention to the second problem now, as multiplication of radicals is the focus of this section. You already have the tools to deal with this problem as well. There is more than one way to think about this problem. For example, we could write it as a square and then use rule 4 to distribute our exponent: $\left(3\sqrt{2}\right)^2 = 3^2\left(\sqrt{2}\right)^2 = 9 \cdot 2 = 18$. (Notice that when you square $\sqrt{2}$, you just get 2. You can think of it simply as the square "undoing" the square root to leave you with the radicand you began with, or you can bring the square inside and take the square root of 4 which still gives you 2.)

There is a more general way to think about this problem (since you might be multiplying two different numbers and hence you would not have a square). First, we need to make sure we understand that multiplication is the operation taking place between the number and the radical even though the symbol is omitted: $3\sqrt{2}$ really means $3 \cdot \sqrt{2}$. Once we realize this, we see that we really just have a multiplication between four numbers here. We know that multiplication is commutative and associative, so we can move things around and multiply any two of them together at a time, so using these properties, we get $3 \cdot \sqrt{2} \cdot 3 \cdot \sqrt{2} = 3 \cdot 3 \cdot \sqrt{2} \cdot \sqrt{2}$ and then grouping together the whole numbers and the radicals, we obtain $9 \cdot 2 = 18$. Notice that we still got the same answer. We just looked at the problem in a more general way. That is part of the beauty of mathematics – as long as you are using correct logic, you will get to the same answer.

There is a nice way to think about this process that is faster (as long as you understand why you can do this) and that is to simply multiply the whole numbers (the outsides) and the radicands (the insides). So very quickly you would get from $3\sqrt{2} \cdot 3\sqrt{2}$ to $9 \cdot 2$ and then finish multiplying….

Now, we are ready for some formal rules for multiplication and division of radicals. These are not really new rules, however. You will see that they are just two of the exponent rules you already know rewritten in radical form!

For non-negative radicands $\boldsymbol{a}$ and $\boldsymbol{b}$,

1. $\boldsymbol{\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}}$

2. $\boldsymbol{\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}}$

The first rule could be written as $a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$, which is really just rule 4 of our exponent rules that says we can distribute exponents over multiplication. The second rule could be written as $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$, which is really just rule 5 of our exponent rules that says we can distribute exponents over division. Here, we are using both of these rules in a backwards kind of way to combine our radicands "under one roof" when we have the *same* root. Some examples will make this very clear.

<u>Examples</u>

Perform the indicated operation and simplify each of the following rational exponent expressions. Assume that all variables represent non-negative real numbers.

1. $\sqrt{7} \cdot \sqrt{5} = \sqrt{7 \cdot 5} = \sqrt{35}$

   We can put our numbers "under the same roof" here since we are multiplying the same type of root. Since we can't pull anything out of the radical, we just go ahead and multiply them together to simplify.

2. $\sqrt{3} \cdot \sqrt{11} = \sqrt{3 \cdot 11} = \sqrt{33}$

For the first two examples, it was convenient to put them under the same radical, but sometimes it is easier to deal with the radicals separately and then multiply your results. This happens when you have perfect powers as in the next example:

3. $\sqrt{25} \cdot \sqrt{16} = 5 \cdot 4 = 20$

Since 25 and 16 are both perfect squares, it is easier to just evaluate the square roots and then multiply your results than it is to put them under the same radical and break them down, although you could do it this way and you would get the same answer:

$$\sqrt{25} \cdot \sqrt{16} = \sqrt{25 \cdot 16} = \sqrt{5 \cdot 5 \cdot 4 \cdot 4} = 5 \cdot 4 = 20$$

4. $\sqrt{5} \cdot \sqrt{5} = \sqrt{5 \cdot 5} = 5$

This problem can be thought of in many ways. For example, $\sqrt{5} \cdot \sqrt{5}$ could also be written as $\left(\sqrt{5}\right)^2$ and then the square "undoes" the square root, or it could be written as $\left(5^{\frac{1}{2}}\right)^2$ and then multiply the exponents to get $5^1 = 5$. You could also bring the square inside the radical $\left(\sqrt{5}\right)^2 = \sqrt{5^2} = \sqrt{25} = 5$. You get the idea, right? As long as you know how radicals operate, you will get to the correct answer.

5. $\sqrt{3} \cdot \sqrt{18} = \sqrt{3 \cdot 18} = \sqrt{3 \cdot 3 \cdot 3 \cdot 2} = 3\sqrt{6}$

Once again, you could have done this problem a different way. If you choose to deal with the radicals separately and then multiply your results, you will get the same answer:

$$\sqrt{3} \cdot \sqrt{18} = \sqrt{3} \cdot \sqrt{2 \cdot 3 \cdot 3} = \sqrt{3} \cdot 3\sqrt{2} = 3\sqrt{3 \cdot 2} = 3\sqrt{6}$$

6. $\sqrt{7x} \cdot \sqrt{21y} = \sqrt{7 \cdot 7 \cdot 3 \cdot x \cdot y} = 7\sqrt{3xy}$

7. $\sqrt{18x^2y^3} \cdot \sqrt{6xy^4} = \sqrt{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3x^3y^7} = 2 \cdot 3xy^3\sqrt{2xy}$

$= 6xy^3\sqrt{2xy}$

Notice that when we put them under the same roof, we automatically break down our numbers and put them in order to make it easier to pull out pairs. Since $18 = 2 \cdot 3 \cdot 3$ and $6 = 2 \cdot 3$ it follows that $18 \cdot 6 = 2 \cdot 3 \cdot 3 \cdot 2 \cdot 3$. Now, just put these primes in order and you will get what is under the radical in the problem above. I recommend doing the factor tree right below the numbers so you can easily put the primes in order in your next step. We used the first rule of exponents $b^m \cdot b^n = b^{m+n}$ to combine the variables under the radical. We used the shortcut learned in the last section to pull out variables (divide the exponent by the index).

8. $\sqrt{14x^5} \cdot \sqrt{21x^6} = \sqrt{2 \cdot 3 \cdot 7 \cdot 7x^{11}} = 7x^5\sqrt{6x}$

9. $3\sqrt{5} \cdot 7\sqrt{10} = 3 \cdot 7 \cdot \sqrt{5} \cdot \sqrt{10} = 21\sqrt{5 \cdot 5 \cdot 2} = 21 \cdot 5\sqrt{2}$

$= 105\sqrt{2}$

This problem could have also been given with parentheses instead of the multiplication symbol like this: $(3\sqrt{5})(7\sqrt{10})$ Oftentimes, the problem is given this way when there are numbers (or variables) in front of the radicals. It means the same thing, just as $5 \cdot 3$ and $(5)(3)$ mean the same thing.

10. $\left(2x\sqrt{x^3y}\right)\left(xy\sqrt{x^5}\right) = 2x^2y\sqrt{x^8y} = 2x^2yx^4\sqrt{y} = 2x^6y\sqrt{y}$

Here again, we multiplied the "outsides" and then put the "insides" under the same roof so that we could simplify. We ended up pulling out x^4 and then combining it with the x^2 that was already in front of the radical.

11. $\left(5mn^2\sqrt{2mn}\right)\left(6n^5\sqrt{6n}\right) = 30mn^7\sqrt{2 \cdot 2 \cdot 3mn^2}$

$= 30 \cdot 2mn^7n\sqrt{3m} = 60mn^8\sqrt{3m}$

12. $-\sqrt[3]{7} \cdot \sqrt[3]{49} = -\sqrt[3]{7 \cdot 7 \cdot 7} = -7$

Don't forget the index represents your exchange rate (how many of something on the inside it takes to equal one of those on the outside). Here the index is 3, so it takes 3 of these 7's to get one of them outside of the radical.

13. $\sqrt[5]{2t} \cdot 4\sqrt[5]{16t^8} = 4\sqrt[5]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2\, t^9} = 4 \cdot 2t\sqrt[5]{t^4} = 8t\sqrt[5]{t^4}$

Sometimes, we are tempted to pull stuff out of the radical when we are not supposed to. Just because we see a square on something doesn't mean it can come out of the square root. Consider the next three examples.

14. $\sqrt{x} \cdot \sqrt{x+1} = \sqrt{x(x+1)}\ or\ \sqrt{x^2+x}$

Even though we have x^2 under the radical, it is attached by addition rather than multiplication, so it cannot come out. In other words, radicals do not distribute over addition (because exponents do not distribute over addition and a radical is just a rational exponent). Consider an example with numbers: $\sqrt{3^2+4^2}$

Is this the same thing as $\sqrt{3^2}+\sqrt{4^2}$? Let's compare them. We know that using the order of operations, we should get the following: $\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$, but $\sqrt{3^2}+\sqrt{4^2} = 3+4 = 7$. So we can see that they are not the same thing. Therefore, $\sqrt{x^2+x} \neq \sqrt{x^2}+\sqrt{x}\ or\ x+\sqrt{x}$, and although it is tempting to do such things, we can see that they produce incorrect results. We need to pay attention at all times in order not to go on autopilot and make these kinds of mistakes.

15. $\sqrt{x+2} \cdot \sqrt{x-2} = \sqrt{(x+2)(x-2)}\ or\ \sqrt{x^2-4}$

Once again, we cannot distribute the radical over addition/subtraction, only over multiplication/division.

16. $\sqrt{y^2(x+y)} \cdot \sqrt{(x+y)^3} = \sqrt{y^2(x+y)^4} = y(x+y)^2$

In this example, we had multiplication inside of the radical, so we were able to take the square root of each factor.

For the next set of examples, we will work with division. These problems are very similar, but you will need to decide when it is convenient to have the radicands under the same radical or separate radicals. We will explore both ways. We are still assuming that all variables represent positive numbers.

17. $\sqrt{\frac{z^2}{16x^2}} = \frac{\sqrt{z^2}}{\sqrt{16x^2}} = \frac{z}{4x}$

In this example, since we could not simplify the radicand by cancelling anything, it was useful to distribute the radical to the numerator and denominator and deal with them separately.

18. $\frac{\sqrt{75y^5}}{\sqrt{3y}} = \sqrt{\frac{75y^5}{3y}} = \sqrt{25y^4} = 5y^2$

In this example, we were given the problem with two separate radicals, but it was convenient to put them "under the same roof" to simplify. Once we simplified, then we dealt with the root. We could have done it a different way, but it would have been more work to deal with them separately:

$$\frac{\sqrt{75y^5}}{\sqrt{3y}} = \frac{\sqrt{3 \cdot 5 \cdot 5y^5}}{\sqrt{3y}} = \frac{5y^2\cancel{\sqrt{3y}}}{\cancel{\sqrt{3y}}} = 5y^2$$

We ended up simplifying at the end instead of the beginning, but we got the same answer!

19. $$\frac{\sqrt{5a^4b^7c^2}}{\sqrt{b^3}} = \sqrt{\frac{5a^4b^7c^2}{b^3}} = \sqrt{5a^4b^4c^2} = a^2b^2c\sqrt{5}$$

It is also useful in this example to put them under the same radical since we see that we can combine our $b's$ together.

20. $$\frac{\sqrt{3x^2y^3}}{4\sqrt{5xy^3}} = \frac{1}{4} \cdot \sqrt{\frac{3x^2y^3}{5xy^3}} = \frac{1}{4} \cdot \sqrt{\frac{3x}{5}} = \frac{\sqrt{3x}}{4\sqrt{5}}$$

First, notice that the 4 in the denominator is outside of the radical, so when we combine our radicals, the 4 needs to stay outside. Also, notice that the 4 is in the denominator, so when we separate it from the radical we need to keep it in the denominator. That is why we have $\frac{1}{4}$ outside being multiplied by our radical. After we put the radicands under the same radical and simplified them, we separated them again at the end and brought the 4 back into the denominator. Later, you will learn how to get rid of the radical in the denominator by rationalizing it.