2.2 Rational Exponents

In section 1.2, we explored the rules of exponents when those exponents were integers. The same rules will apply, however, if we have powers which are rational numbers (fractions with integer numerator and denominator). This section will review those rules, but also will introduce the meaning of a rational exponent.

The key to understanding rational exponents is simply a definition:

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m or \sqrt[n]{b^m}$$

Notice that the denominator becomes the index of the radical and the numerator becomes the power.

Either form will give the same answer when you simplify, but I believe the first form is easier most of the time (since it is easier to take a root of a smaller number), so you will see this one used more often in this text. Sometimes it is more convenient to bring the power inside of the root, but only when there are leftovers inside of the radical.

<u>Examples</u>

Simplify each of the following rational exponent expressions.

1. $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

   So 8 to the $\frac{2}{3}$ power is just 4. Notice that we first took the cube root of 8 and then we squared our result. Working from the inside out makes it easy because we can focus on one thing at a time. We could have also used the second form and brought the exponent inside before taking the root: $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$. Either way, we will get 4.

2. $16^{\frac{1}{2}} = \left(\sqrt[2]{16}\right)^1 = \sqrt{16} = 4$

You can see here that the power of $\frac{1}{2}$ just means square root.

3. $4^{\frac{5}{2}} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

Here we take the square root of 4 first and then put it to the fifth power. Don't forget that $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 8 \cdot 4 = 32$. You can group the 2's together in whichever way you like since the 5 together rather than to multiply five 2's together, so be careful to pay attention to what you are doing. The more practice you get, the less likely you are to make such mistakes.

4. $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$

If you did not know that $\sqrt[3]{125} = 5$, then you could simply factor 125 using a factor tree. You will recognize quickly that 5 divides 125 (the divisibility rule for 5 says that if a number ends in a 0 or a 5 that it is divisible by 5). Then $125 = 5 \cdot 25$ and the rest is easy….

5. $(-27)^{\frac{4}{3}} = \left(\sqrt[3]{-27}\right)^4 = \left(-\sqrt[3]{27}\right)^4 = (-3)^4 = 81$

We can pull the negative out of the cube root, but we still have to apply the fourth power to it, which gives us positive 81.

6. $(-16)^{\frac{3}{4}} = \left(\sqrt[4]{-16}\right)^3$ ➪ Not a real number

This is not a real number because we cannot take an even root of a negative (there is no real number that when put to the fourth power will yield a negative number).

7. $-16^{\frac{3}{4}} = -\left(\sqrt[4]{16}\right)^3 = -2^3 = -8$

In this example, the negative sign is outside of the radical because the exponent is only attached to the number 16 (there are no parentheses around -16).

In the next example, we end up with a number left inside of the radical. All of the examples we have done so far have been perfect roots with nothing left inside. There is more than one way to approach this problem.

8. $54^{\frac{4}{3}} = \left(\sqrt[3]{54}\right)^4 = \left(3\sqrt[3]{2}\right)^4 = 3^4\left(\sqrt[3]{2}\right)^4 = 81\sqrt[3]{2^4}$

Notice that the power of 4 was distributed to both 3 and $\sqrt[3]{2}$ in the third step using the exponent rule $(a \cdot b)^n = a^n \cdot b^n$ from section 1.2. The power was then brought inside of the radical using the second form of the definition.

$= 81\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2} = 81 \cdot 2\sqrt[3]{2} = 162\sqrt[3]{2}$

Here is another way using the second form of the definition in the beginning:

$$54^{\frac{4}{3}} = \left(\sqrt[3]{54}\right)^4 = \sqrt[3]{54^4} = \sqrt[3]{54 \cdot 54 \cdot 54 \cdot 54} = 54\sqrt[3]{54}$$

$$= 54\sqrt[3]{3 \cdot 3 \cdot 3 \cdot 2} = 54 \cdot 3\sqrt[3]{2} = 162\sqrt[3]{2}$$

Either way you approach the problem, you will get the same answer.

The exponent rules still apply when the exponents are rational numbers. We will now go through each of these rules and provide an example.

We will assume that all variables represent positive real numbers for the examples in this section.

Rule	Example
1. $b^m \cdot b^n = b^{m+n}$	$x^{\frac{2}{3}} \cdot x^{\frac{4}{3}} = x^{\frac{2}{3}+\frac{4}{3}} = x^{\frac{6}{3}} = x^2$
2. $\frac{b^m}{b^n} = b^{m-n}$	$\frac{y^{\frac{5}{9}}}{y^{\frac{1}{4}}} = y^{\frac{5}{9}-\frac{1}{4}} = y^{\frac{20}{36}-\frac{9}{36}} = y^{\frac{11}{36}}$
3. $(b^m)^n = b^{mn}$	$\left(x^{\frac{4}{3}}\right)^{\frac{2}{3}} = x^{\frac{4}{3}\cdot\frac{2}{3}} = x^{\frac{8}{9}}$

Don’t forget that Rule 1 and Rule 3 often get confused, so make sure to note whether you will be adding your powers or multiplying your powers. In rule 3, we have a single base while in Rule 1, we have two of the same base. This might help you to remember.

4. $(a \cdot b)^m = a^m \cdot b^m$	$\left(27x^9y^{\frac{1}{5}}\right)^{\frac{2}{3}} = 27^{\frac{2}{3}}(x^9)^{\frac{2}{3}}\left(y^{\frac{1}{5}}\right)^{\frac{2}{3}}$ $= \left(\sqrt[3]{27}\right)^2 x^{\frac{9}{1}\cdot\frac{2}{3}} y^{\frac{1}{5}\cdot\frac{2}{3}} = 9x^6y^{\frac{2}{15}}$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ | $\left(\frac{x}{36y^8}\right)^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{(36y^8)^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{36^{\frac{1}{2}}(y^8)^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{6y^4}$

6. $b^{-n} = \frac{1}{b^n}$ | $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{4}$

7. $b^0 = 1$ | $\left(59x^{\frac{4}{7}}y^{\frac{3}{5}}z\right)^0 = 1$

The next set of examples will integrate these rules. A good rule of thumb is to work from the outside in when you have parentheses involved. Look at the outermost exponent first and if it is negative, move that piece across the fraction bar using rule 6. When all of the outermost exponents are positive, then distribute exponents over multiplication using rules 3 and 4. Then combine exponents using rules 1 and 2.

Examples

Simplify each of the following rational exponent expressions. Assume all variables represent non-negative real numbers.

1. $(25b^4)^{\frac{1}{2}} = 25^{\frac{1}{2}}(b^4)^{\frac{1}{2}} = \sqrt{25}b^2 = 5b^2$

For this problem, the outermost exponent is positive, so we can begin by distributing the exponent.

2. $(-27x^6y^{-2})^{\frac{1}{3}} = \sqrt[3]{-27}(x^6)^{\frac{1}{3}}(y^{-2})^{\frac{1}{3}} = -3x^2y^{-\frac{2}{3}} = -\frac{3x^2}{y^{\frac{2}{3}}}$

Note that you could have approached this problem differently by working on the inside first (moving y^{-2} down) and then applying the outer exponent. You will get the same answer. We will always approach the problems the same way (by working from the outside in) in order to provide a consistent technique, but there is usually more than one way to get the answer. As long as you are following the rules, you will get there. Also, you could write your answer with radicals: $-\frac{3x^2}{\sqrt[3]{y^2}}$ In general, however, if the problem is given with exponents, the answer is expected to have exponents. Similarly, if the problem is given with radicals, then the answer is expected to have radicals.

3. $\left(\frac{-8x^3}{27y^{-6}}\right)^{-\frac{2}{3}} = \left(\frac{27y^{-6}}{-8x^3}\right)^{\frac{2}{3}} = \frac{(27)^{\frac{2}{3}}\left(y^{-6}\right)^{\frac{2}{3}}}{(-8)^{\frac{2}{3}}\left(x^3\right)^{\frac{2}{3}}} = \frac{\left(\sqrt[3]{27}\right)^2 y^{-4}}{\left(\sqrt[3]{-8}\right)^2 x^2} = \frac{9}{4x^2y^4}$

When the entire fraction is put to a negative power, you can flip the fraction to make the outer exponent positive, but be careful not to change anything inside when you flip it. Once again, you could have started by moving y^{-6} up and then flipping.

4. $\left(25x^{-\frac{1}{3}}y^2\right)^{-\frac{1}{2}}\left(16x^5y^{-\frac{1}{3}}\right)^{\frac{3}{4}} = \frac{\left(16x^5y^{-\frac{1}{3}}\right)^{\frac{3}{4}}}{\left(25x^{-\frac{1}{3}}y^2\right)^{\frac{1}{2}}} = \frac{8x^{\frac{15}{4}}y^{-\frac{1}{4}}}{5x^{-\frac{1}{6}}y}$

$$= \frac{8x^{\frac{15}{4}}x^{\frac{1}{6}}}{5y^{\frac{1}{4}}y} \quad = \frac{8x^{\frac{47}{12}}}{5y^{\frac{5}{4}}}$$

This is a good problem to test your understanding of working with rational exponents.

5. $\left[(x+2)^{\frac{1}{8}}\right]^{8} = x + 2$

This problem reminds us that radicals and powers "undo" each other since inside we have the 8th root and outside we have the power 8. It is easier to just use rule 3 and multiply the powers together to get a power of 1.

6. $y^{\frac{2}{3}}\left(y^{-\frac{2}{3}} + y^{\frac{1}{2}}\right) = y^{\frac{2}{3}} \cdot y^{-\frac{2}{3}} + y^{\frac{2}{3}} \cdot y^{\frac{1}{2}} = y^{0} + y^{\frac{7}{6}} = 1 + y^{\frac{7}{6}}$

The answer to this one could also be written with radicals as

$1 + \sqrt[6]{y^{7}}\ or\ 1 + y\sqrt[6]{y}$