2.1 Simplifying Radicals

A radical symbol looks like this: $\sqrt{}$

This symbol means the *positive square root* (often called the *principal square root*).

Every positive real number actually has two square roots, but the symbol above just asks for the positive one.

> For example, $2^2 = 4$, but can you think of another number that you can square that will also give you 4? What about -2?
>
> It is true that $(-2)^2 = 4$ as well. This means that 4 must have two square roots, both 2 and -2. Even though there are two square roots, the principal square root is 2 (i.e. $\sqrt{4} = 2$).

Definition: A real number 'c' is called a **square root** of a real number 'a' if $a = c^2$. The **principal square root** is given by $\sqrt{a} = |c|$.

Note: We will see later that when we are solving equations like $x^2 = 4$, we need to give both square roots as answers since we want to know what x can be that will make the equation true. This means that when we take the square root of both sides to "undo" the square, we will need to attach a ± symbol to give us both answers because the radical only gives us the positive answer, but more on that later. We can easily see that we should get both answers if we use the factoring method to solve:

$$x^2 = 4$$

⇨ $x^2 - 4 = 0$

⇨ $(x + 2)(x - 2) = 0$

⇨ $x + 2 = 0\,;\ x - 2 = 0$

⇨ $x = -2; x = 2.$

<u>Examples</u>

Simplify each of the following radical expressions.

1. $\sqrt{25} = 5$ since $5^2 = 25$
2. $\sqrt{49} = 7$ since $7^2 = 49$
3. $\sqrt{16} = 4$ since $4^2 = 16$

Notice that the examples above are quite simple and we know them quickly from our multiplication tables, but sometimes the number inside of the radical (called the **radicand**) is not so clean and nice. For that reason, we could really use a technique that will work to simplify any radical. We will begin by thinking about the easy examples above to introduce this technique, and then we can move on to more interesting examples.

We could think of the radical symbol as an exchange rate between the inside and the outside of the radical. Since we are dealing with a *square* root, two of something on the inside will equal one of that same thing on the outside. (For a *cube* root, you will see the exchange rate becomes three meaning that three of something on the inside equals one of them on the outside and for a *fourth* root, it will be four, etc.)

Consider example 1 above to illustrate this idea:

If we factor the inside of the radical all the way and then pair up our factors, we can pull out one for every pair:

$$\sqrt{25} = \sqrt{5 \cdot 5} = 5\sqrt{1} = 5 \cdot 1 = 5$$

Notice that we had a pair of 5's inside, so we were able to pull out a single 5 (Two 5's on the inside equals one 5 on the outside). After we pulled out 5, we only had 1 left inside, but the principal square root of 1 is just 1. When our numbers are small perfect squares like 25, there is no need to do all of this, but it does illustrate the *process* nicely. When the number inside of the radical is large or is not a perfect square, this process is a nice way to simplify, but it does depend on a working knowledge of basic multiplication facts and divisibility rules in order to break numbers down into their prime decompositions. Let's use this process on examples 2 and 3 above as well just to make sure we understand it:

$\sqrt{49} = \sqrt{7 \cdot 7} = 7\sqrt{1} = 7$ (Circling the pairs often helps to keep track of what numbers are being pulled out).

$\sqrt{16} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2} = 2 \cdot 2\sqrt{1} = 4$

(Note that you must multiply the numbers that you pull outside of the radical in order to completely simplify. Also, you don't need to write $\sqrt{1}$ if you understand what is going on here. We will omit this step in future examples.) Also, you could have broken 16 down into $4 \cdot 4$ and then pulled out 4. Our examples in this section will always break the numbers down into primes, but sometimes you can stop early if you see a pair. Let's do some more examples.

4. $-\sqrt{36} = -\sqrt{2 \cdot 2 \cdot 3 \cdot 3} = -2 \cdot 3 = -6$

(The negative sitting outside of the radical is just a -1 waiting to be multiplied by your result. The multiplication happens *after* the radical is evaluated because the radical is really an exponent as you will see in the next section. Exponents come before multiplication in the order of operations.)

What if the negative is inside of the radical instead of outside, though?

5. $\sqrt{-36}$ is *not a real number* because there is no real number that you can multiply by itself an get a negative. This is because a positive times a positive is positive and a negative times a negative is also positive. There is no way to get a negative when squaring a

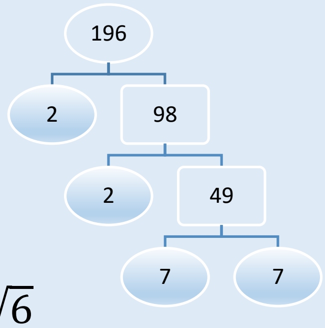
number. (You will see in section 2.7 that this does have an answer, but it is imaginary: $\sqrt{-36} = 6i$. For now, we just say this is "*not a real number*" and we leave it at that.)

For larger radicands, a factor tree is a helpful tool to break it down into primes. We will demonstrate in the next two examples.

6. $\sqrt{196} = \sqrt{2 \cdot 2 \cdot 7 \cdot 7} = 2 \cdot 7 = 14$

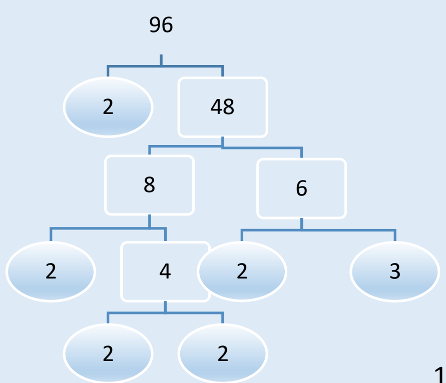
For this one, you may need to do a factor tree to get the prime factorization and your divisibility rules will come in handy here. You know that 2 divides any even number, so $196 = 2$ times something, so to figure out what that something is, simply divide it by 2. Then you will see that $196 = 2 \cdot 98$. Continue to break down 98 in the same way….

Circling the primes as you go makes it easy to find them.



7. $\sqrt{96} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 2 \cdot 2\sqrt{6} = 4\sqrt{6}$

One possible factor tree for 96 is on the below. You can break the radicand down in many ways, but the prime factorization will be the same. The primes are circled. They are put in increasing order under the radical for easy pairing.



Factor trees will be omitted for the upcoming examples as this is prerequisite knowledge, but please make sure you understand where the numbers under the radicals are coming from by doing the factor tree for yourself.

Now that we have explored square roots, we will turn our attention to cube roots, fourth roots, fifth roots, etc. The type of root is called the *index* and you can find the index at the top left corner outside of the radical.

For example, a cube root has a 3 as its index: $\sqrt[3]{}$

Instead of pairing up factors, we will triple them up, so that for every three of something on the inside we get one of them on the outside. The index gives us the exchange rate. If the index is 4, then we need four of something on the inside to give us one of them on the outside.

The index is omitted for square roots (nobody ever puts a 2 outside of the square root) because the meaning of the radical symbol is understood.

8. $\sqrt[3]{96} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 2\sqrt[3]{12}$

9. $\sqrt[3]{-16} = -\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2} = -2\sqrt[3]{2}$

Notice that we can take the cubed root of a negative number by simply pulling the negative outside. This is because if you cube a negative number, you will get a negative number.

For example, $(-2)^3 = (-2)(-2)(-2) = -8$.

This will be true for any odd root (3, 5, 7, 9, etc.) since any negative number to an odd power will produce a negative number.

Therefore, if you have an odd root, you may simply pull the negative outside. Be careful, though, since you cannot do this for an even root (2,4,6,8, etc.). You must pay close attention to the index.

I wonder what would happen if we allowed variables into our examples……

We all know that variables typically represent numbers in mathematics, so we should treat these variables just like we treat the numbers. We just have to be careful about things that we don't know – like whether the variable represents a positive or a negative number since this could affect our even roots. If we assume that all of our variables represent positive numbers, then the situation is simple, just like working with the numbers. There is even a little shortcut to deal with the variables that will be revealed after you have an understanding of where this shortcut comes from and why it works.

10. $\sqrt{8x^3y^5} = \sqrt{2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y} = 2xy^2\sqrt{2xy}$

Notice here that we wrote out what x^3 and y^5 actually mean. Notice also that we treat them the same way we treat the numbers. Here we are assuming that our variables represent positive numbers.

(If we were not assuming that our variables were positive, we would need to place absolute value symbols around x because if x were indeed negative, then $x \cdot x$ would still be positive so we could take the square root, but it would be the principal square root. Just pulling out x is not enough because the principal square root is the positive square root, so $|x|$ would denote the positive square root. More on this to come.)

11. $\sqrt{54x^9y^3z^2}$

$$= \sqrt{2 \cdot 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot z \cdot z}$$

You can see that if we break down the variables this way, it might get very tedious when the exponents are large. There is a simpler way to do this, though. Notice that when we pair them up, we are really diving the exponent by 2. You can see that for x^9, we have 4 pairs with 1 left over. When you divide 9 by 2, don't you get 4 with remainder 1? The shortcut for variables is simply to divide the exponent by the index. This tells you how many you have on the outside and your remainder is how many are left inside. So here we should have x^4 on the outside and x^1, or just x, left on the inside. Similarly for y^3, just divide the exponent 3 by the index 2 and you see that you have a single y on the outside and one of them left on the inside. The final answer should be:

$$= 3x^4yz\sqrt{6xy}.$$

Again, we are still assuming that all variables are positive in this example.

What if we do not make that assumption? In the next set of examples, we will allow our variables to represent any real number (which means they could be negative). You will need to pay close attention to the instructions for problems such as this because the instructions tell you what the assumptions are about the variables.

12. $\sqrt{9x^2} = \sqrt{3 \cdot 3 \cdot x^2} = 3|x|$

We need absolute value symbols around x here because if it is negative, then we will not be taking the principal square root without these symbols. Imagine that x is -1 to see that this is true. Go through the calculation above and see what would happen if the absolute value symbol was not there and every x was replaced by -1.

13. $\sqrt[4]{80x^{20}y^{43}z^2} = \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot x^{20}y^{43}z^2}$ = (see below)

This is an even root, so we do need to have absolute values on our variables as we pull them out if we are not sure the result comes out positive. (If it has an even power when you pull it out, you don't really need the absolute value since it will automatically be positive).

Since the index is 4, we need four of anything to give us one of them outside of the radical. The power on x is 20, and if we divide 20 by 4, we get 5 with no remainder, so we should have x^5 outside of the radical with no $x's$left inside. If x is negative, then x^5 will also be negative, so we do need absolute value symbols around x. (It doesn't matter if the power is inside or outside of the absolute value symbols.)

The power on y is 43, and if we divide 43 by 4, we get 10 with remainder 3, so we should have y^{10} outside of the radical with y^3 left on the inside. If y is negative, then y^{10} will still be positive, so we do NOT need absolute value symbols around y, although it is ok to put them there.

The power on z is 2, and we do not have enough of them to pull any out at all.

$$= 2|x^5| \cdot |y^{10}| \cdot \sqrt[4]{5y^3z^2}$$

$$= 2|x|^5y^{10}\,\sqrt[4]{5y^3z^2}$$

14. $\sqrt[3]{64x^6y^9} = \sqrt[3]{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot x^6y^9} = 2\cdot 2x^2y^3 = 4x^2y^3$

Notice that we do not need to worry about absolute value symbols when we have an odd root. This is because the negative can just come out along with the number. If a variable is negative, then the cubed root of that variable will also be negative.

15. $\sqrt{(x-1)^2} = |x-1|$

You can think of the square root as "undoing" the square if you like, but you will need the absolute value symbols since you do not know if $x-1$ is positive or negative.

16. $\sqrt{x^2+8x+16} = \sqrt{(x+4)(x+4)} = |x+4|$

Be careful not to distribute the radical over addition here. You must factor the inside just as you do with numbers.

17. $\sqrt{36x^2-120x+100} = \sqrt{4(9x^2-30x+25)}$

We must factor this one a little bit more than the others. It is easier if you pull the GCF out first and then continue to factor.

$$= \sqrt{2\cdot 2\cdot (3x-5)(3x-5)} = |2(3x-5)| = 2|3x-5|$$

Note: In the last step, we simplified using the following property of absolute values: $|a\cdot b| = |a|\cdot|b|$

Reminder: As you work the practice set, please make sure to pay attention to the instructions so you will know whether you need to include the absolute value symbol in your answers or not.