1.13 Polynomial Division

When we want a model for any operation in algebra, we can always go back to what we know about arithmetic. We have one more basic operation to explore with polynomials – division. To divide polynomials, we will build on the same processes that work with numbers, once again. You will use the process of dividing polynomials to graph rational functions in your next course and beyond, so it is a very important skill to have before you get there!

For division by monomials, there is a simpler process than long division that can be applied, so we will begin with this. For division by binomials or larger polynomials, long division will be necessary, although sometimes you will be able to use a shortcut called "synthetic division" (namely, when you are dividing by a linear polynomial of the form *x*-*c*). We will talk about that later in this section.

Dividing by monomials

We begin with an arithmetic example and then consider algebraic examples. We know we can do the following arithmetic example in two ways and arrive at the same value:

Example

$$\underbrace{(45+10)}_{55} \div 5 = 55 \div 5 = 11$$

An alternative way (that works well when the numerator has unlike terms).

$$\begin{aligned} \frac{45+10}{5} &= \frac{45}{5} + \frac{10}{5} \\ &= 9 + 2 \\ &= 11 \end{aligned}$$

The same idea applies to algebraic examples.

Algebraic Examples (for monomials)

1. $(2x^3 + 6x^2 + 4x) \div (2x)$

$$\frac{2x^3+6x^2+4x}{2x} = \frac{2x^3}{2x} + \frac{6x^2}{2x} + \frac{4x}{2x}$$

$$= x^2 + 3x + 2$$

We could have factored out the GCF and cancelled:

$$\frac{(2x^3+6x^2+4x)}{2x} = \frac{2x(x^2+3x+2)}{2x} = x^2 + 3x + 2$$

2. $(2x^4y^6 - 3x^3y^4 + 5x^2y^3) \div (2x^2y^2)$

$$\frac{2x^4y^6-3x^3y^4+5x^2y^3}{2x^2y^2} = \frac{2x^4y^6}{2x^2y^2} - \frac{3x^3y^4}{2x^2y^2} + \frac{5x^2y^3}{2x^2y^2}$$

$$= x^2y^4 - \frac{3xy^2}{2} + \frac{5y}{2}$$

The prior examples involved dividing by a monomial. What if we are dividing by a binomial or larger?

We can use long division!

Polynomial Long Division

Recall the process (in arithmetic):

$$8541 \div 32 \text{ or } \frac{8541}{32}$$

$$\begin{array}{r}
266\frac{29}{32} \\
32\overline{)8541} \\
\underline{-64} \\
214 \\
\underline{-192} \\
221 \\
\underline{-192} \\
29
\end{array}$$

Algebraic Examples (binomials and bigger)

1. $\frac{x^2+5x+6}{x+2}$ or $(x^2+5x+6) \div (x+2)$

Note: For this one, you <u>could</u> just factor the numerator and cancel.

$$\frac{x^2+5x+6}{x+2} = \frac{\cancel{(x+2)}(x+3)}{\cancel{x+2}} = x+3$$

Note: Distributing doesn't help us here!

$$\frac{x^2+5x+6}{x+2} = \underbrace{\frac{x^2}{x+2} + \frac{5x}{x+2} + \frac{6}{x+2}}_{\text{None of these can be simplified since we cannot cancel over addition.}}$$

If we were to do long division:

Step 1

$$\begin{array}{r|l} & \;\; x \\ \cline{2-2} x+2 & x^2+5x+6 \end{array}$$

The first step is the "division" step. You will be using the x from the divisor $x + 2$ and the first term of the dividend $x^2 + 5x + 6$ which is x^2. You are asking yourself x times what will give you x^2? Since $x \cdot x = x^2$, we write x directly over x^2.

Step 2

$$\begin{array}{r|l} & \;\; x \\ \cline{2-2} x+2 & x^2+5x+6 \\ & x^2+2x \end{array}$$

$x(x+2)$

The second step is the "multiplication" step. We will multiply $x(x + 2)$ and write the answer directly beneath $x^2 + 5x + 6$. (Note: $x(x + 2)$ comes from the x above the division symbol and the $x + 2$ which precedes the division symbol).

Step 3

$$\begin{array}{r|l} & \;\; x \\ \cline{2-2} x+2 & x^2+5x+6 \\ & -(x^2+2x) \\ \cline{2-2} \end{array}$$

The third step is the "subtraction" step. We will distribute the negative (change the signs) and subtract.

$$
\begin{array}{r|l}
 & \quad x \\
\hline
x+2 & x^2+5x+6 \\
 & \underline{-x^2-2x \quad \downarrow} \\
 & \qquad\quad 3x+6
\end{array}
$$

Bring down the next term and start the process over again, if necessary.

<u>Repeat steps 1 thru 3</u>

$$
\begin{array}{r|l}
 & \quad x \\
\hline
x+2 & x^2+5x+6 \\
 & \underline{-x^2-2x \quad \downarrow} \\
 & \qquad 3x+6
\end{array}
$$

In the division step, we still use the x from the divisor $x+2$, but now we use the $3x$ from the bottom line $3x+6$. Now, x times what is $3x$? Since x times 3 is $3x$, we will write $+3$ on top next to the x.

$$
\begin{array}{r|l}
 & \quad x+3 \\
\hline
x+2 & x^2+5x+6 \\
 & \underline{-x^2-2x \quad \downarrow} \\
 & \qquad 3x+6
\end{array}
$$

Next, multiply.

$$
\begin{array}{r|l}
 & \quad x+3 \\
\hline
x+2 & x^2+5x+6 \\
 & \underline{-x^2-2x \quad \downarrow} \\
 & \qquad 3x+6 \\
 & \qquad 3x+6
\end{array}
$$

$3\ (x+2)$

$$\begin{array}{r|l} & \quad x+3 \\ \hline x+2 & x^2+5x+6 \\ & -x^2-2x \downarrow \\ \hline & \quad 3x+6 \\ & -(3x+6) \end{array}$$

Now, subtract.

Distribute negative!

$$\begin{array}{r|l} & \quad x+3 \\ \hline x+2 & x^2+5x+6 \\ & -x^2-2x \downarrow \\ \hline & \quad 3x+6 \\ & -3x-6 \\ \hline & \qquad 0 \end{array}$$

Now, subtract.

So, the quotient (answer) is $x+3$.

2. $\frac{x^2+7x+10}{x+3}$

Step 1 - Divide

$$\begin{array}{r|l} & x \\ \hline x+3 & x^2+7x+10 \end{array}$$

Step 2 - Multiply

$$\begin{array}{r|l} & x \\ \hline x+3 & x^2+7x+10 \\ & x^2+3x \end{array}$$

Step 3 - Subtract

$$\begin{array}{r|l} & x \\ \hline x+3 & x^2+7x+10 \\ & -(x^2+3x) \end{array}$$

$$\begin{array}{r|l} & x \\ \hline x+3 & x^2+7x+10 \\ & \underline{-x^2-3x \quad \downarrow} \\ & \qquad\quad 4x+10 \end{array}$$

Repeat Steps 1 thru 3

Step 1 - Divide

$$\begin{array}{r|l} & x+4 \\ \hline x+3 & x^2+7x+10 \\ & \underline{-x^2-3x \quad \downarrow} \\ & \qquad\quad 4x+10 \end{array}$$

Step 2 - Multiply

$$\begin{array}{r|l} & x+4 \\ \hline x+3 & x^2+7x+10 \\ & \underline{-x^2-3x \quad \downarrow} \\ & \qquad\quad 4x+10 \\ & \qquad\quad 4\,(x+3) \end{array}$$

$$\begin{array}{rl} & \quad x+4 \\ x+3 & \overline{\big)\, x^2+7x+10} \\ & -x^2-3x \quad \downarrow \\ \hline & \qquad 4x+10 \\ & \qquad 4x+12 \end{array}$$

Step 3 - Subtract

$$\begin{array}{rl} & \quad x+4 \\ x+3 & \overline{\big)\, x^2+7x+10} \\ & -x^2-3x \quad \downarrow \\ \hline & \qquad 4x+10 \\ & \quad -(4x+12) \end{array}$$

$$\begin{array}{rl} & \quad x+4 \\ x+3 & \overline{\big)\, x^2+7x+10} \\ & -x^2-3x \quad \downarrow \\ \hline & \qquad 4x+10 \\ & \quad\ -4x-12 \\ \hline & \qquad\quad -2 \end{array}$$

The quotient is $x+4-\frac{2}{x+3}$.

3. $(2x^2+3x-4) \div (x-2)$

Step 1 - Divide

$$\begin{array}{rl} & \quad 2x \\ x-2 & \overline{\big)\, 2x^2+3x-4} \end{array}$$

Step 2 - Multiply

$$
\begin{array}{r l}
 & 2x \\
x-2 & \overline{\big)\,2x^2+3x-4} \\
 & \;\;2x^2-4x
\end{array}
$$

Step 3 - Subtract

$$
\begin{array}{r l}
 & 2x \\
x-2 & \overline{\big)\,2x^2+3x-4} \\
 & -(2x^2-4x)
\end{array}
$$

$$
\begin{array}{r l}
 & 2x \\
x-2 & \overline{\big)\,2x^2+3x-4} \\
 & \underline{-2x^2+4x\quad\;\;\downarrow} \\
 & \qquad\quad\;\; 7x-4
\end{array}
$$

Repeat Steps 1 thru 3
Step 1 - Divide

$$
\begin{array}{r l}
 & 2x+7 \\
x-2 & \overline{\big)\,2x^2+3x-4} \\
 & \underline{-2x^2+4x\quad\;\;\downarrow} \\
 & \qquad\quad\;\; 7x-4
\end{array}
$$

Step 2 - Multiply

$$
\begin{array}{r l}
 & 2x+7 \\
x-2 & \overline{\big)\,2x^2+3x-4} \\
 & \underline{-2x^2+4x\quad\;\;\downarrow} \\
 & \qquad\quad\;\; 7x-4 \\
 & \qquad\quad\;\; 7x-14
\end{array}
$$

Step 3 - Subtract

$$
\begin{array}{r l}
 & \quad 2x + 7 \\
x - 2 & \overline{\big)\, 2x^2 + 3x - 4} \\
 & \underline{-\, 2x^2 + 4x \qquad} \\
 & \qquad\quad 7x - 4 \\
 & \qquad\ \underline{-7x + 14} \\
 & \qquad\qquad\quad 10
\end{array}
$$

The quotient is $2x + 7 + \frac{10}{x-2}$.

4. $\frac{6x^3+11x^2-19x-2}{3x-2}$

Step 1 - Divide

$$
\begin{array}{r l}
 & \quad 2x^2 \\
3x - 2 & \overline{\big)\, 6x^3 + 11x^2 - 19x - 2}
\end{array}
$$

Step 2 - Multiply

$$
\begin{array}{r l}
 & \quad 2x^2 \\
3x - 2 & \overline{\big)\, 6x^3 + 11x^2 - 19x - 2} \\
 & \ \ 6x^3 - \ 4x^2
\end{array}
$$

Step 3 - Subtract

$$
\begin{array}{r l}
 & \quad 2x^2 \\
3x - 2 & \overline{\big)\, 6x^3 + 11x^2 - 19x - 2} \\
 & \underline{-\, 6x^3 + \ 4x^2 \qquad\quad} \\
 & \qquad\quad 15x^2 - 19x
\end{array}
$$

Repeat Steps 1 thru 3

Step 1 - Divide

$$\begin{array}{rl} & 2x^2+5x \\ 3x-2 & \overline{\big)\,6x^3+11x^2-19x-2} \\ & \underline{-\,6x^3+\ 4x^2\qquad\qquad} \\ & \qquad\quad 15x^2-19x \end{array}$$

Step 2 - Multiply

$$\begin{array}{rl} & 2x^2+5x \\ 3x-2 & \overline{\big)\,6x^3+11x^2-19x-2} \\ & \underline{-\,6x^3+\ 4x^2\qquad\qquad} \\ & \qquad\quad 15x^2-19x \\ & \qquad\quad 15x^2-10x \end{array}$$

Step 3 - Subtract

$$\begin{array}{rl} & 2x^2+5x \\ 3x-2 & \overline{\big)\,6x^3+11x^2-19x-2} \\ & \underline{-\,6x^3+\ 4x^2\qquad\qquad} \\ & \qquad\quad 15x^2-19x \\ & \underline{\qquad -15x^2+10x} \\ & \qquad\qquad\qquad -\ 9x-2 \end{array}$$

Repeat Steps 1 thru 3

Step 1 - Divide

$$\begin{array}{rl} & 2x^2+5x-3 \\ 3x-2 & \overline{\big)\,6x^3+11x^2-19x-2} \\ & \underline{-\,6x^3+\ 4x^2\qquad\qquad} \\ & \qquad\quad 15x^2-19x \\ & \underline{\qquad -15x^2+10x} \\ & \qquad\qquad\qquad -\ 9x-2 \end{array}$$

Step 2 - Multiply

$$
\begin{array}{r r}
 & 2x^2 + 5x - 3 \\
3x - 2 & \overline{\big)\, 6x^3 + 11x^2 - 19x - 2} \\
 & \underline{-\,6x^3 + \;4x^2 \qquad\qquad\quad} \\
 & 15x^2 - 19x \qquad \\
 & \underline{-15x^2 + 10x \qquad} \\
 & -\;9x - 2 \\
 & -\,9x + 6
\end{array}
$$

Step 3 - Subtract

$$
\begin{array}{r r}
 & 2x^2 + 5x - 3 \\
3x - 2 & \overline{\big)\, 6x^3 + 11x^2 - 19x - 2} \\
 & \underline{-\,6x^3 + \;4x^2 \qquad\qquad\quad} \\
 & 15x^2 - 19x \qquad \\
 & \underline{-15x^2 + 10x \qquad} \\
 & -\;9x - 2 \\
 & \underline{+\,9x \;-\; 6} \\
 & -\,8
\end{array}
$$

The quotient is $2x^2 + 5x - 3 - \frac{8}{3x-2}$.

5. $(2x^4 + 3x^3 - 3x^2 - 5x - 3) \div (2x^2 - x - 1)$

$$
\begin{array}{r|l}
 & x^2 + 2x \\
\hline
2x^2 - x - 1 & 2x^4 + 3x^3 - 3x^2 - 5x - 3 \\
 & \underline{-(2x^4 - x^3 - x^2)} \\
 & \qquad 4x^3 - 2x^2 - 5x \\
 & \quad \underline{-(4x^3 - 2x^2 - 2x)} \\
 & \qquad\qquad\qquad -3x - 3 \;=\; -(3x + 3)
\end{array}
$$

The quotient is $x^2 + 2x + \frac{-3x-3}{2x^2-x-1}$ or $x^2 + 2x - \frac{3x+3}{2x^2-x-1}$.

6. $(2x^3 + 5x^2 - 1) \div (x - 1)$

Rewrite $2x^3 + 5x^2 - 1$ filling in the "missing" terms which will act as placeholders: $(2x^3 + 5x^2 + 0x - 1) \div (x - 1)$.

$$
\begin{array}{r|l}
 & 2x^2 + 7x + 7 \\
\hline
x - 1 & 2x^3 + 5x^2 + 0x - 1 \\
 & \underline{-(2x^3 - 2x^2)} \\
 & \qquad 7x^2 + 0x \\
 & \quad \underline{-(7x^2 - 7x)} \\
 & \qquad\qquad 7x - 1 \\
 & \qquad\quad \underline{-(7x - 7)} \\
 & \qquad\qquad\qquad 6
\end{array}
$$

The quotient is $2x^2 + 7x + 7 + \frac{6}{x-1}$.

7. $(x^3 - 1) \div (x - 1)$

We could rewrite this as a fraction, then factor and cancel, if possible.

$\frac{x^3-1}{x-1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = x^2 + x + 1$ or we can use long division.

First, we will rewrite $x^3 - 1$ inserting the "missing" terms.

$(x^3 + 0x^2 + 0x - 1) \div (x - 1)$

$$
\begin{array}{rl}
 & x^2 + x + 1 \\
x - 1 & \overline{\big)\, x^3 + 0x^2 + 0x - 1} \\
 & \underline{-(x^3 - x^2)} \\
 & \qquad\quad x^2 + 0x \\
 & \qquad \underline{-(x^2 - x)} \\
 & \qquad\qquad\quad x - 1 \\
 & \qquad\qquad \underline{-(x - 1)} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

The quotient is $x^2 + x + 1$.

<u>Synthetic Division</u>

This is a "shortcut" method for polynomial division, but it is only used when you are dividing by something linear that has the form $x - c$.

<u>Examples</u>

1. $(2x^3 - 9x^2 + 10x - 7) \div (x - 3)$

Let's take a look at the solution using long division.

$$
\begin{array}{r@{}l}
 & \quad 2x^2 - 3x + 1 \\
x - 3 & \overline{\big)\, 2x^3 - 9x^2 + 10x - 7} \\
 & \underline{-(2x^3 - 6x^2)} \\
 & \qquad\quad -3x^2 + 10x \\
 & \qquad \underline{-(-3x^2 + 9x)} \\
 & \qquad\qquad\qquad\quad x - 7 \\
 & \qquad\qquad\qquad \underline{-(x - 3)} \\
 & \qquad\qquad\qquad\qquad -4
\end{array}
$$

The quotient is $2x^2 - 3x + 1 - \frac{4}{x-3}$.

Now, let’s use synthetic division.

$(2x^3 - 9x^2 + 10x - 7) \div (x - 3)$

The coefficients 2, -9, 10, and -7 will be placed inside the symbol used for synthetic division. The first coefficient, 2, is dropped down as indicated by the arrow. Outside of the symbol will be the solution to $x - 3 = 0$ which is 3. As you can see, no variables are included during synthetic division.

$$
\begin{array}{r|rrrr}
 & 2 & -9 & 10 & -7 \\
 & \downarrow & & & \\
3 & & & & \\
\hline
 & 2 & & &
\end{array}
$$

Next, multiply 3 and 2 (the “outside” numbers) and write the answer, 6, inside the symbol in the next column.

$$\begin{array}{r|rrrr} & 2 & -9 & 10 & -7 \\ 3 & \downarrow & 6 & & \\ \hline & 2 & & & \end{array}$$

Now, add the second column, $-9 + 6$, and write the answer below the line.

$$\begin{array}{r|rrrr} & 2 & -9 & 10 & -7 \\ 3 & \downarrow & 6 & & \\ \hline & 2 & -3 & & \end{array}$$

We continue by repeating the process of multiplication followed by addition. So, we multiply $3\,(-3)$ and write the answer in the third column below 10.

$$\begin{array}{r|rrrr} & 2 & -9 & 10 & -7 \\ 3 & \downarrow & 6 & -9 & \\ \hline & 2 & -3 & & \end{array}$$

Now, add the third column and write the answer below the line.

$$\begin{array}{r|rrrr} & 2 & -9 & 10 & -7 \\ 3 & \downarrow & 6 & -9 & \\ \hline & 2 & -3 & 1 & \end{array}$$

Continue alternating between multiplication followed by addition until you run out of columns.

$$\begin{array}{r|rrrr} & 2 & -9 & 10 & -7 \\ 3 & \downarrow & 6 & -9 & 3 \\ \hline & 2 & -3 & 1 & -4 \end{array}$$

The final step is to write out the answer. The numbers below the line (the last row) are the coefficients of your answer. Using the same variable as the original problem, decrease the power one for the first term $(2x^2)$. Decrease the power by one again for the second term $(-3x)$, and so on until you get to the last number in the "answer" row. The last number in the answer row is the numerator of the remainder.

So, the quotient is $2x^2 - 3x + 1 - \frac{4}{x-3}$.

2. $(2x^4 + x^2 - 1) \div (x + 1)$

First, rewrite the polynomial $2x^4 + x^2 - 1$ including any "missing" terms. So, $2x^4 + x^2 - 1 = 2x^4 + 0x^3 + x^2 + 0x - 1$. You must include the zeros in the first row.

$$
\begin{array}{r|rrrrr}
 & 2 & 0 & 1 & 0 & -1 \\
-1 & & -2 & 2 & -3 & 3 \\
\hline
 & 2 & -2 & 3 & -3 & 2
\end{array}
$$

So, the quotient is $2x^3 - 2x^2 + 3x - 3 + \frac{2}{x+1}$.

3. $(x^3 - 3x^2 + 7x - 10) \div (x - 2)$

$$
\begin{array}{r|rrrr}
 & 1 & -3 & 7 & -10 \\
2 & & 2 & -2 & 10 \\
\hline
 & 1 & -1 & 5 & 0
\end{array}
$$

So, the quotient is $x^2 - x + 5$.